# Determinant Representations of Correlation Functions for the Supersymmetric $t$-$J$ Model

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**Abstract**

Working in the $F$-basis provided by the factorizing $F$-matrix, the scalar products of Bethe states for the supersymmetric $t$-$J$ model are represented by determinants. By means of these results, we obtain determinant representations of correlation functions for the model.

# 1 Introduction

The computation of correlation functions is one of the major challenging problems in the theory of quantum integrable models [1, 2]. There are currently two approaches for computing the correlation functions of a quantum integrable model. One is the vertex operator method (see e.g. [3-8]), and another one is based on the detailed analysis of the structure of the Beth states [9, 10].

Progress has recently been made in the literature on the second approach with the help of the Drinfeld twists. Working in the $F$-bases provided by the $F$-matrices (Drinfeld twists), the authors in [11, 12] managed to compute the form factors and correlation functions of the XXX and XXZ models analytically and expressed them in compact determinant forms.

Recently we have constructed the Drinfeld twists for both the rational $gl(m|n)$ and the quantum $U_q(gl(m|n))$ supersymmetric models and resolved the hierarchy of their nested Bethe vectors in the $F$-basis [13, 14, 15]. In [16], we obtained the determinant representation of the scalar products and the correlation functions for the $U_q(gl(1|1))$ free fermion model.

Quantum integrable models associated with Lie superalgebras [17, 18, 19] are physically important because they give strongly correlated fermion models of superconductivity. Among them, the $t$-$J$ model with the Hamiltonian given by

$$
\begin{aligned}
H &= -t \sum_{j=1}^{N} \sum_{\sigma=\uparrow,\downarrow} [c_{j,\sigma}^{\dagger}(1-n_{j,-\sigma})c_{j+1,\sigma}(1-n_{j+1,-\sigma}) + c_{j+1,\sigma}^{\dagger}(1-n_{j+1,-\sigma})c_{j+1,\sigma}(1-n_{j,-\sigma})] \\
&\quad + J \sum_{j=1}^{N} [S_j^z S_{j+1}^z + \frac{1}{2}(S_j^{\dagger}S_{j+1} + S_j S_{j+1}^{\dagger}) - \frac{1}{4}n_j n_{j+1}]
\end{aligned}
\tag{1.1}
$$

was proposed in an attempt to understand high-$T_c$ superconductivity [20, 21, 22, 23]. It is a strongly correlated electron system with nearest-neighbor hopping ($t$) and anti-ferromagnetic exchange ($J$) of electrons. When $J = 2t$, the $t$-$J$ model becomes $gl(2|1)$ invariant (i.e. supersymmetric). Using the nested algebraic Bethe ansatz method, Essler and Korepin obtained the eigenvalues of the supersymmetric $t$-$J$ model [24]. The algebraic structure and physical properties of the model were investigated in [25, 26].

In this paper, using our previous results in [13] and [15], we give the determinant representation of the scalar products and the correlation functions of the supersymmetric $t$-$J$ model. In section 2, we review the background of the supersymmetric $t$-$J$ model and its algebraic Bethe ansatz. In section 3, we apply our results on the Drinfeld twists to construct

the determinant representations of the components of the Bethe states. In section 4, we obtain the determinant representation of the scalar products of the Bethe states. Then in section 5, we compute the correlation functions of the local fermion operators of the model. We conclude the paper by offering some discussions in section 6.

# 2  The supersymmetry $t$-$J$ model

## 2.1  Some background of the model

Let $V$ be the 3-dimensional $gl(2|1)$-module and $R \in End(V \otimes V)$ the $R$-matrix associated with this module. $V$ is $Z_2$-graded, and in the following we choose the FFB grading for $V$, i.e. $[1] = [2] = 1, [3] = 0$. The $R$-matrix depends on the difference of two spectral parameters $\lambda_1$ and $\lambda_2$ associated with the two copies of $V$, and is, in the FFB grading, given by

$$R_{12}(\lambda_1, \lambda_2) \equiv R_{12}(\lambda_1 - \lambda_2) = \begin{pmatrix} c_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{12} & 0 & -b_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{12} & 0 & 0 & 0 & b_{12} & 0 & 0 \\ 0 & -b_{12} & 0 & a_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{12} & 0 & b_{12} & 0 \\ 0 & 0 & b_{12} & 0 & 0 & 0 & a_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{12} & 0 & a_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

(2.1)

where

$$a_{12} = a(\lambda_1, \lambda_2) \equiv \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2 + \eta}, \qquad b_{12} = b(\lambda_1, \lambda_2) \equiv \frac{\eta}{\lambda_1 - \lambda_2 + \eta},$$
$$c_{12} = c(\lambda_1, \lambda_2) \equiv \frac{\lambda_1 - \lambda_2 - \eta}{\lambda_1 - \lambda_2 + \eta}$$

(2.2)

with $\eta \in C$ being the crossing parameter. One can easily check that the $R$-matrix satisfies the unitary relation

$$R_{21} R_{12} = 1.$$

(2.3)

Here and throughout $R_{ij} \equiv R_{ij}(\lambda_i, \lambda_j)$. The $R$-matrix satisfies the graded Yang-Baxter equation (GYBE)

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.$$

(2.4)

3

In terms of the matrix elements defined by

$$R(\lambda)(v^{i'} \otimes v^{j'}) = \sum_{i,j} R(\lambda)_{ij}^{i'j'}(v^i \otimes v^j),$$ (2.5)

the GYBE reads

$$\sum_{i',j',k'} R(\lambda_1 - \lambda_2)_{ij}^{i'j'} R(\lambda_1 - \lambda_3)_{i'k}^{i''k'} R(\lambda_2 - \lambda_3)_{j'k'}^{j''k''} (-1)^{[j'] ([i'] + [i''])}$$

$$= \sum_{i',j',k'} R(\lambda_2 - \lambda_3)_{jk}^{j'k'} R(\lambda_1 - \lambda_3)_{ik'}^{i'k''} R(\lambda_1 - \lambda_2)_{i'j'}^{i''j''} (-1)^{[j']([i]+[i'])}.$$ (2.6)

The quantum monodromy matrix $T(\lambda)$ of the supersymmetric $t$-$J$ model on a lattice of length $N$ is defined as

$$T(\lambda) = R_{0N}(\lambda, \xi_N) R_{0N-1}(\lambda, \xi_{N-1})...R_{01}(\lambda, \xi_1),$$ (2.7)

where the index 0 refers to the auxiliary space and $\{\xi_i\}$ are arbitrary inhomogeneous parameters depending on site $i$. $T(\lambda)$ can be represented in the auxiliary space as the $3 \times 3$ matrix whose elements are operators acting on the quantum space $V^{\otimes N}$:

$$T(\lambda) = \begin{pmatrix} A_{11}(\lambda) & A_{12}(\lambda) & B_1(\lambda) \\ A_{21}(\lambda) & A_{22}(\lambda) & B_2(\lambda) \\ C_1(\lambda) & C_2(\lambda) & D(\lambda) \end{pmatrix}_{(0)}.$$ (2.8)

By using the GYBE, one may prove that the monodromy matrix satisfies the GYBE

$$R_{12}(\lambda - \mu)T_1(\lambda)T_2(\mu) = T_2(\mu)T_1(\lambda)R_{12}(\lambda - \mu).$$ (2.9)

or in matrix form,

$$\sum_{i',j'} R(\lambda - \mu)_{ij}^{i'j'} T(\lambda)_{i'}^{i''} T(\mu)_{j'}^{j''} (-1)^{[j']([i']+[i''])}$$

$$= \sum_{i',j'} T(\mu)_j^{j'} T(\lambda)_i^{i'} R(\lambda - \mu)_{i'j'}^{i''j''} (-1)^{[j']([i]+[i'])}.$$ (2.10)

Define the transfer matrix $t(\lambda)$

$$t(\lambda) = str_0 T(\lambda),$$ (2.11)

where $str_0$ denotes the supertrace over the auxiliary space. With the help of the GYBE, one may check that the transfer matrix satisfies the commutation relation $[t(\lambda), t(\mu)] = 0$, ensuring the integrability of the system.

The transfer matrix gives the Hamiltonian of the system:

$$H = \frac{\partial \ln t(\lambda)}{\partial \lambda}|_{\lambda=0}$$

$$= -\sum_{j=1}^{N} \left\{ \sum_{\sigma=\uparrow,\downarrow} (Q_{j+1,\sigma}^{\dagger} Q_{j,\sigma} + Q_{j,\sigma}^{\dagger} Q_{j+1,\sigma}) - 2S_j^z S_{j+1}^z - S_j S_{j+1}^{\dagger} - S_{j+1} S_j^{\dagger} + 2T_j T_{j+1} \right\},$$

(2.12)

where $Q_\sigma, Q_\sigma^{\dagger}, S, S^{\dagger}, S^z, T$ are generators of the superalgebra $gl(2|1)$. The fundamental representations of these operators take the following form

$$S_j^z = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_j = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_k = e_{21}^k, \quad S_k^{\dagger} = e_{12}^k,$$

$$Q_{k,\uparrow} = e_{32}^k, \quad Q_{k,\uparrow}^{\dagger} = e_{23}^k, \quad Q_{k,\downarrow} = e_{31}^k, \quad Q_{k,\downarrow}^{\dagger} = e_{13}^k,$$

(2.13)

where $e_{ij}^k$ is a $3 \times 3$ matrix acting on the $k$-th space with elements $(e_{ij}^k)_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta}$. Using the standard fermion representation

$$S_j = c_{j,\uparrow}^{\dagger} c_{j,\downarrow}, \quad S_j^{\dagger} = c_{j,\downarrow}^{\dagger} c_{j,\uparrow}, \quad S_j^z = \frac{1}{2}(n_{j,\uparrow} - n_{j,\downarrow})$$

$$Q_{j,\sigma} = (1 - n_{j,-\sigma}) c_{j,\sigma}, \quad Q_{j,\sigma}^{\dagger} = (1 - n_{j,-\sigma}) c_{j,\sigma}^{\dagger}, \quad T_j = 1 - \frac{1}{2} n_j,$$

$$n_{j,\sigma} = c_{j,\sigma}^{\dagger} c_{j,\sigma}, \quad n_j = n_{j,\uparrow} + n_{j,\downarrow},$$

(2.14)

one finds that (2.12) gives the Hamiltonian (1.1) at the supersymmetric point $J = 2t$.

## 2.2 Algebraic Bethe ansatz

The transfer matrix (2.11) can be diagonalized by using the nested algebraic Bethe ansatz. The Bethe state of the supersymmetric $t$-$J$ model is defined as follows.

**Definition 1** *Let $|0\rangle$ be the pseudo-vacuum state of the quantum tensor space $V^{\otimes N}$, and $|0\rangle^{(1)}$ be the pseudo-nested-vacuum state of the nested quantum tensor space $(V^{(1)})^{\otimes n}$, i.e.,*

$$|0\rangle = \otimes_{i=1}^{N} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{(i)}, \quad |0\rangle^{(1)} = \otimes_{j=1}^{n} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{(j)}.$$

(2.15)

*The Bethe state of the supersymmetric $t$-$J$ model is then defined by*

$$|\Omega_N(\{\lambda_j\})\rangle = \sum_{d_1 \dots d_n} (\Omega_n^{(1)})^{d_1 \dots d_n} C_{d_1}(\lambda_1) \dots C_{d_n}(\lambda_n) |0\rangle \quad (\lambda_1 \neq \dots \neq \lambda_n),$$

(2.16)

5

where $d_i = 1, 2$, $(\Omega_n^{(1)})^{d_1 \ldots d_n}$ is a component of the nested Bethe state $|\Omega\rangle^{(1)}$ via

$$|\Omega_n(\{\lambda_j^{(1)}\})\rangle^{(1)} = C^{(1)}(\lambda_1^{(1)}) \cdots C^{(1)}(\lambda_m^{(1)})|0\rangle^{(1)} \qquad (\lambda_1^{(1)} \neq \ldots \neq \lambda_m^{(1)}), \qquad (2.17)$$

and $C^{(1)}$, the creation operator of the nested $gl(2)$ system, is the lower-triangular entry of the nested monodromy matrix $T^{(1)}$

$$
\begin{aligned}
T^{(1)}(\lambda^{(1)}) &= r_{0n}(\lambda^{(1)} - \lambda_n)r_{0n-1}(\lambda^{(1)} - \lambda_{n-1}) \ldots r_{01}(\lambda^{(1)} - \lambda_1) \\
&\equiv \begin{pmatrix} A^{(1)}(\lambda^{(1)}) & B^{(1)}(\lambda^{(1)}) \\ C^{(1)}(\lambda^{(1)}) & D^{(1)}(\lambda^{(1)}) \end{pmatrix}_{(0)}
\end{aligned} \qquad (2.18)
$$

with

$$r_{12}(\lambda_1, \lambda_2) \equiv r_{12}(\lambda_1 - \lambda_2) = \begin{pmatrix} c_{12} & 0 & 0 & 0 \\ 0 & a_{12} & -b_{12} & 0 \\ 0 & -b_{12} & a_{12} & 0 \\ 0 & 0 & 0 & c_{12} \end{pmatrix}. \qquad (2.19)$$

Similarly, we can also define the dual Bethe state $\langle \Omega_N |$.

**Definition 2** *With the help of the dual pseudo-vacuum state $\langle 0 |$ and the dual pseudo-nested-vacuum state $\langle 0 |^{(1)}$, the dual Bethe state is defined by*

$$\langle \Omega_N(\{\mu_j\})| = \sum_{f_n, \ldots, f_1} (\Omega^{(1)})^{f_n \ldots f_1} \langle 0| B_{f_n}(\mu_n) \ldots B_{f_1}(\mu_1) \qquad (\mu_n \neq \ldots \neq \mu_1), \qquad (2.20)$$

*where $(\Omega^{(1)})^{f_n \ldots f_1}$ is a component of the dual nested Bethe state $\langle \Omega |^{(1)}$*

$$\langle \Omega_n(\{\mu_j^{(1)}\})|^{(1)} = \langle 0|^{(1)} B^{(1)}(\mu_m^{(1)}) \ldots B^{(1)}(\mu_1^{(1)}) \qquad (\mu_m^{(1)} \neq \ldots \neq \mu_1^{(1)}). \qquad (2.21)$$

The diagonalization of the transfer matrix $t(\lambda)$ (2.11) leads to the following theorem [24]:

**Theorem 1** *The Bethe states $|\Omega_N(\{\lambda_j\})\rangle$ defined by (2.16) are eigenstates of the transfer matrix $t(\lambda)$ if the spectral parameters $\lambda_j$ ($j = 1, \ldots, n$) satisfy the Bethe ansatz equations (BAE)*

$$\prod_{k=1}^{N} a(\lambda_j, \xi_k) \prod_{l=1}^{m} a^{-1}(\lambda_j, \lambda_l^{(1)}) = 1 \qquad (j = 1, \ldots, n) \qquad (2.22)$$

*and the nested Bethe ansatz equations (NBAE)*

$$\prod_{j=1}^{n} a(\lambda_j, \lambda_l^{(1)}) \prod_{k=1, \neq l}^{m} \frac{\lambda_k^{(1)} - \lambda_l^{(1)} + \eta}{\lambda_k^{(1)} - \lambda_l^{(1)} - \eta} = 1 \qquad (l = 1, \ldots, m). \qquad (2.23)$$

*The eigenvalues $\Lambda(\lambda, \{\lambda_k\}, \{\lambda_j^{(1)}\})$ of the transfer matrix $t(\lambda)$ are given by*

$$\Lambda(\lambda, \{\lambda_k\}, \{\lambda_j^{(1)}\})$$
$$= \prod_{i=1}^{N} a(\lambda, \xi_i) \prod_{j=1}^{n} \frac{1}{a(\lambda, \lambda_j)} \Lambda^{(1)}(\lambda) + \prod_{j=1}^{n} \frac{1}{a(\lambda_j, \lambda)}, \tag{2.24}$$

*where $\Lambda^{(1)}(\lambda)$ is the eigenvalues of the nested transfer matrix $t^{(1)}(\lambda) = str_0 T^{(1)}(\lambda)$*

$$\Lambda^{(1)}(\lambda) = -\prod_{j=1}^{n} \frac{a(\lambda, \lambda_j)}{a(\lambda_j, \lambda)} \prod_{k=1}^{m} \frac{1}{a(\lambda, \lambda_k^{(1)})} - \prod_{j=1}^{n} a(\lambda, \lambda_j) \prod_{k=1}^{m} \frac{1}{a(\lambda^{(1)}, \lambda)}. \tag{2.25}$$

One easily checks that this theorem also holds for the dual Bethe state $\langle \Omega_N(\{\mu_j\})|$ defined by (2.20) if we change the spectral parameters $\lambda_j$ and $\lambda_j^{(1)}$ in (2.22)-(2.25) to $\mu_j$ and $\mu_j^{(1)}$, respectively.

# 3 Symmetric representations of the Bethe state

## 3.1 Factorizing $F$-matrix and its inverse

With the help of the permutation group $\sigma \in \mathcal{S}_N$, one may introduce the $R$-matrix $R_{1...N}^{\sigma}$ [11, 27] which can be expressed in terms of the elementary $R$-matrix $R_{i\,i+1} = R_{1...N}^{\sigma_i}$ for any elementary permutation $\sigma_i(i, i+1) = (i+1, i)$ through the decomposition law $R_{1...N}^{\sigma'\sigma} = R_{\sigma'(1...N)}^{\sigma} R_{1...N}^{\sigma'}$. We proved in [13, 14, 15] that for the $R$-matrix $R_{1...N}^{\sigma}$, there exists a non-degenerate lower-diagonal $F$-matrix satisfying the relation

$$F_{\sigma(1...N)}(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(N)}) R_{1...N}^{\sigma}(\xi_1, \ldots, \xi_N) = F_{1...N}(\xi_1, \ldots, \xi_N). \tag{3.1}$$

Explicitly,

$$F_{1...N} = \sum_{\sigma \in \mathcal{S}_N} \sum_{\alpha_{\sigma(1)} \cdots \alpha_{\sigma(N)}}^{*} \prod_{j=1}^{N} P_{\sigma(j)}^{\alpha_{\sigma(j)}} S(c, \sigma, \alpha_{\sigma}) R_{1...N}^{\sigma}, \tag{3.2}$$

where the sum $\sum^*$ is over all non-decreasing sequences of the labels $\alpha_{\sigma(i)}$:

$$\alpha_{\sigma(i+1)} \geq \alpha_{\sigma(i)}, \quad \text{if} \quad \sigma(i+1) > \sigma(i),$$
$$\alpha_{\sigma(i+1)} > \alpha_{\sigma(i)}, \quad \text{if} \quad \sigma(i+1) < \sigma(i) \tag{3.3}$$

and the c-number function $S(c, \sigma, \alpha_{\sigma})$ is given by

$$S(c, \sigma, \alpha_{\sigma}) \equiv \exp \left\{ \frac{1}{2} \sum_{l>k=1}^{N} \left(1 - (-1)^{[\alpha_{\sigma(k)}]}\right) \delta_{\alpha_{\sigma(k)}, \alpha_{\sigma(l)}} \ln(1 + c_{\sigma(k)\sigma(l)}) \right\}. \tag{3.4}$$

The inverse of the $F$-matrix is given by

$$F_{1\ldots N}^{-1} = F_{1\ldots N}^* \prod_{i<j} \Delta_{ij}^{-1} \tag{3.5}$$

with

$$[\Delta_{ij}]_{\alpha_i\alpha_j}^{\beta_i\beta_j} = \delta_{\alpha_i\beta_i}\delta_{\alpha_j\beta_j} \begin{cases} a_{ij} & \text{if } \alpha_i > \alpha_j \\ a_{ji} & \text{if } \alpha_i < \alpha_j \\ 1 & \text{if } \alpha_i = \alpha_j = 3 \\ 4a_{ij}a_{ji} & \text{if } \alpha_i = \alpha_j = 1,2 \end{cases} . \tag{3.6}$$

and

$$F_{1\ldots N}^* = \sum_{\sigma\in\mathcal{S}_N} \sum_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}}^{**} S(c,\sigma,\alpha_\sigma) R_{\sigma(1\ldots N)}^{\sigma^{-1}} \prod_{j=1}^{N} P_{\sigma(j)}^{\alpha_{\sigma(j)}}, \tag{3.7}$$

where the sum $\sum^{**}$ is taken over all possible $\alpha_i$ which satisfies the following non-increasing constraints:

$$\begin{aligned} \alpha_{\sigma(i+1)} &\leq \alpha_{\sigma(i)}, & \text{if} \quad \sigma(i+1) < \sigma(i), \\ \alpha_{\sigma(i+1)} &< \alpha_{\sigma(i)}, & \text{if} \quad \sigma(i+1) > \sigma(i). \end{aligned} \tag{3.8}$$

## 3.2 Bethe state in the $F$-basis

The non-degeneracy of the $F$-matrix means that its column vectors form a complete basis, which is called the $F$-basis. In [15, 13], we found that in the $F$-basis, the creation and annihilation operators $C_i(\lambda)$ and $B_i(\lambda)$ $(i=1,2)$ of the supersymmetrix $t$-$J$ model have the symmetric form:

$$\tilde{C}_2(\lambda) = F_{1\ldots N}C_2(\lambda)F_{1\ldots N}^{-1} = \sum_{i=1}^{N} b_{0i}E_{(i)}^{23} \otimes_{j\neq i} \text{diag}\,(a_{0j}, 2a_{0j}, 1)_{(j)}, \tag{3.9}$$

$$\tilde{C}_1(u) = F_{1\ldots N}C_1(\lambda)F_{1\ldots N}^{-1} = \sum_{i=1}^{N} b_{0i}E_{(i)}^{23} \otimes_{j\neq i} \text{diag}\,\left(2a_{0j}, a_{0j}a_{ij}^{-1}, 1\right)_{(j)}$$

$$- \sum_{i\neq j}^{N} \frac{\eta b_{0j}a_{0i}}{\xi_i - \xi_j} E_{(i)}^{12} \otimes E_{(j)}^{23} \otimes_{k\neq i,j} \text{diag}\,\left(2a_{0k}, a_{0k}a_{ik}^{-1}, 1\right)_{(k)}, \tag{3.10}$$

$$\tilde{B}_2(\lambda) = F_{1\ldots N}B_2(\lambda)F_{1\ldots N}^{-1} = -\sum_{i=1}^{N} b_{0i}E_{(i)}^{32} \otimes_{j\neq i} \text{diag}\,\left(a_{0j}, a_{0j}(2a_{ji})^{-1}, a_{ji}^{-1}\right)_{(j)}, \tag{3.11}$$

8

$$\tilde{B}_1(\lambda) = F_{1...N} B_1(\lambda) F_{1...N}^{-1} = -\sum_{i=1}^{N} b_{0i} E_{(i)}^{31} \otimes_{j\neq i} \text{diag}\left(a_{0j}(2a_{ji})^{-1}, a_{0j}a_{ji}^{-1}, a_{ji}^{-1}\right)_{(j)}$$

$$-\sum_{i\neq j}^{N} \frac{\eta b_{0i} a_{0j}}{\xi_j - \xi_i} E_{(i)}^{32} \otimes E_{(j)}^{21} \otimes_{k\neq i,j} \text{diag}\left(a_{0k}(2a_{kj})^{-1}, a_{0k}a_{ki}^{-1}, a_{ki}^{-1}\right)_{(k)}, \qquad (3.12)$$

where $a_{0j} \equiv a(\lambda, \xi_j)$ $b_{0j} \equiv b(\lambda, \xi_j)$.

Acting the associated $F$-matrix on the pseudo-vacuum state $|0\rangle$, one finds that the pseudo-vacuum state is invariant. It is due to the fact that only the term with all roots equal to 3 will produce non-zero results. Therefore, the $gl(2|1)$ Bethe state (2.16) in the $F$-basis can be written as

$$|\tilde{\Omega}_N(\{\lambda_j\})\rangle \equiv F_{1...N}|\Omega_N(\{\lambda_j\})\rangle$$
$$= \sum_{d_1...d_n} (\Omega_n^{(1)})^{d_1...d_n} \tilde{C}_{d_1}(\lambda_1)...\tilde{C}_{d_n}(\lambda_n)|0\rangle. \qquad (3.13)$$

Without loss of generality, we will only concentrate on the Bethe state with the **quantum number** $p$ which indicates the number of $d_i = 2$, and will use the notation $|\tilde{\Omega}_N(\{\lambda_j\}_{(p,n)})\rangle$ with the subscript pair $(p,n)$ to denote a Bethe state which has quantum number $p$ and has $n$ spectral parameters.

**Proposition 1** *The Bethe state of the supersymmetric t-J model can be represented in the $F$-basis by*

$$|\tilde{\Omega}_N(\{\lambda_j\}_{(p,n)})\rangle$$
$$= \sum_{\sigma\in\mathcal{S}_N} Y_R(\{\lambda_{\sigma(i)}\}, \{\lambda_{\sigma(j)}^{(1)}\}) \tilde{C}_2(\lambda_{\sigma(1)})...\tilde{C}_2(\lambda_{\sigma(p)})\tilde{C}_1(\lambda_{\sigma(p+1)})...\tilde{C}_1(\lambda_{\sigma(n)})|0\rangle \qquad (3.14)$$

$$= \sum_{\sigma\in\mathcal{S}_N} Y_R(\{\lambda_{\sigma(i)}\}, \{\lambda_{\sigma(j)}^{(1)}\}) \sum_{i_1<...<i_p} \sum_{i_{p+1}<...<i_n} 2^{\frac{p(p-1)+(n-p)(n-p-1)}{2}} \prod_{l=1}^{p} \prod_{k=p+1}^{n} a(\lambda_{\sigma(l)}, \xi_{i_k})$$
$$\times \det\mathcal{B}_p(\lambda_{\sigma(1)}, ..., \lambda_{\sigma(p)}; \xi_{i_1}, ..., \xi_{i_p})$$
$$\times \det\mathcal{B}_{n-p}(\lambda_{\sigma(p+1)}, ..., \lambda_{\sigma(n)}; \xi_{i_{p+1}}, ..., \xi_{i_n}) \overrightarrow{\prod_{j=1}^{p} E_{(i_j)}^{23}} \overrightarrow{\prod_{j=p+1}^{n} E_{(i_j)}^{13}} |0\rangle \qquad (3.15)$$

*with the sets $\{i_1, ..., i_p\} \cap \{i_{p+1}, ..., i_n\} = \varnothing$ and the prefactor $Y_R$ being*

$$Y_R(\{\lambda_{\sigma(i)}\}, \{\lambda_{\sigma(j)}^{(1)}\})$$
$$= \frac{1}{p!(n-p)!} c_{1...n}^\sigma B_{n-p}^*\left(\lambda_{p+1}^{(1)}, ..., \lambda_n^{(1)}|\lambda_{\sigma(p+1)}, ..., \lambda_{\sigma(n)}\right) \prod_{k=p+1}^{n} \prod_{l=1}^{p} \left(-\frac{2a(\lambda_{\sigma(k)}, \lambda_{\sigma(l)})}{a(\lambda_{\sigma(l)}, \lambda_{\sigma(k)})}\right).$$
$$(3.16)$$

Here $c_{1,\ldots,n}^{\sigma}$ has the decomposition law $c_{1\ldots n}^{\sigma'\sigma} = c_{\sigma'(1\ldots n)}^{\sigma} c_{1\ldots n}^{\sigma'}$ with $c_{1\ldots n}^{\sigma_i} = c_{i\ i+1} \equiv c(v_i, v_{i+1})$ for an elementary permutation $\sigma_i$, the c-number $B_p^*$ is given by

$$B_p^*\left(\lambda_1^{(1)}, \ldots, \lambda_p^{(1)} | \lambda_1, \ldots, \lambda_p\right)$$

$$= \sum_{\sigma \in \mathcal{S}_p} \prod_{k=1}^{p} \left(-b(\lambda_k^{(1)}, \lambda_{\sigma(k)})\right) \prod_{j \neq \sigma(k), \ldots, \sigma(p)} \frac{c(\lambda_k^{(1)}, \lambda_j)}{2a(\lambda_{\sigma(k)}, \lambda_j)} \prod_{l=k+1}^{p} 2a(\lambda_k^{(1)}, \lambda_{\sigma(l)}),$$

and the elements of the $n \times n$ matrix $\mathcal{B}_n(\{\lambda_i\}; \{\xi_j\})$ are

$$(\mathcal{B}_n)_{\alpha\beta} = b(\lambda_\alpha, \xi_\beta) \prod_{\gamma=1}^{\alpha-1} a(\lambda_\gamma, \xi_\beta). \tag{3.17}$$

In (3.15), we have used the convention $\overrightarrow{\prod_{i=1}^{n}} f_i \equiv f_1 \ldots f_n$. For our later use, we also introduce the notation $\overleftarrow{\prod_{i=1}^{n}} f_i \equiv f_n \ldots f_1$.

*Proof.* By using the exchange symmetry of the Bethe state

$$|\tilde{\Omega}_N(\{\lambda_{\sigma(j)}\})\rangle = \frac{1}{c_{1\ldots n}^{\sigma}}|\tilde{\Omega}_N(\{\lambda_j\})\rangle, \ \sigma \in \mathcal{S}_n \tag{3.18}$$

and the commutation relation between $C_i(\lambda)$ and $C_j(\mu)$

$$\tilde{C}_i(\mu)\tilde{C}_j(\lambda) = -\frac{1}{a(\lambda, \mu)}\tilde{C}_j(\lambda)\tilde{C}_i(\mu) + \frac{b(\lambda, \mu)}{a(\lambda, \mu)}\tilde{C}_j(\mu)\tilde{C}_i(\lambda), \tag{3.19}$$

we have showed in [13, 15] that the Bethe state (3.13) can be written as

$$|\tilde{\Omega}_N(\{\lambda_j\}_{(p,n)})\rangle = \sum_{\sigma \in \mathcal{S}_n} Y_R(\{\lambda_{\sigma(i)}\}, \{\lambda_{\sigma(j)}^{(1)}\}) \sum_{i_1 < \ldots < i_p} \sum_{i_{p+1} < \ldots < i_n} \prod_{k=p+1}^{n} \prod_{l=1}^{p} a(\lambda_{\sigma(l)}, \xi_{i_k})$$

$$\times B_p(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(p)} | \xi_{i_1}, \ldots, \xi_{i_p})$$

$$\times B_{n-p}(\lambda_{\sigma(p+1)}, \ldots, \lambda_{\sigma(n)} | \xi_{i_{p+1}}, \ldots, \xi_{i_n}) \overrightarrow{\prod_{j=1}^{p}} E_{(i_j)}^{23} \overrightarrow{\prod_{j=p+1}^{n}} E_{(i_j)}^{13} |0\rangle, \tag{3.20}$$

where

$$B_n(\lambda_1, \ldots, \lambda_n | \xi_{i_1}, \ldots, \xi_{i_n}) = \sum_{\sigma \in \mathcal{S}_n} \text{sign}(\sigma) \prod_{k=1}^{n} b(\lambda_k, \xi_{i_{\sigma(k)}}) \prod_{l=k+1}^{n} 2a(\lambda_k, \xi_{i_{\sigma(l)}}). \tag{3.21}$$

One checks that the function $B_n(\lambda_1, \ldots, \lambda_n | \xi_{i_1}, \ldots, \xi_{i_n})$ is equivalent to the determinant $2^{n(n-1)/2} \det \mathcal{B}_n(\{\lambda_k\}, \{\xi_j\})$, thus proving the proposition.

10

$$\square$$

By a similar procedure, one may prove the following proposition for the dual Bethe state $\langle \tilde{\Omega}_N(\{\mu_j\}_{(p,n)})|$ (2.20):

**Proposition 2** *The dual Bethe state $\langle \tilde{\Omega}_N(\{\mu_j\}_{(p,n)})|$ of the supersymmetric t-J model can be represented by*

$$
\begin{aligned}
\langle \tilde{\Omega}_N(\{\mu_j\}_{(p,n)})| &= \sum_{\sigma \in \mathcal{S}_N} Y_L(\{\mu_{\sigma(i)}\}, \{\mu^{(1)}_{\sigma(j)}\}) \langle 0| \tilde{B}_1(\mu_{\sigma(n)}) \ldots \tilde{B}_1(\mu_{\sigma(p+1)}) \\
&\quad \times \tilde{B}_2(\mu_{\sigma(p)}) \ldots \tilde{B}_2(\mu_{\sigma(1)}) \qquad\qquad\qquad (3.22) \\
&= \sum_{\sigma \in \mathcal{S}_N} Y_L(\{\mu_{\sigma(i)}\}, \{\mu^{(1)}_{\sigma(j)}\}) \sum_{i_1 < \ldots < i_p} \sum_{i_{p+1} < \ldots < i_n} (-1)^n 2^{-\frac{p(p-1)+(n-p)(n-p-1)}{2}} \\
&\quad \times \prod_{l=1}^{p} \prod_{k=p+1}^{n} a(\mu_{\sigma(l)}, \xi_{i_k}) \prod_{l=1}^{p} \prod_{k \neq i_l, i_{p+1}, \ldots, i_n}^{N} a^{-1}(\xi_k, \xi_{i_l}) \prod_{l=p+1}^{n} \prod_{k=1, \neq i_l}^{N} a^{-1}(\xi_k, \xi_{i_l}) \\
&\quad \times \det \mathcal{B}_p(\mu_{\sigma(1)}, \ldots, \mu_{\sigma(p)}; \xi_{i_1}, \ldots, \xi_{i_p}) \\
&\quad \times \det \mathcal{B}_{n-p}(\mu_{\sigma(p+1)}, \ldots, \mu_{\sigma(n)}; \xi_{i_{p+1}}, \ldots, \xi_{i_n}) \langle 0| \overset{\overleftarrow{n}}{\prod_{j=p+1}} E^{31}_{(i_j)} \overset{\overleftarrow{p}}{\prod_{j=1}} E^{32}_{(i_j)}, \qquad (3.23)
\end{aligned}
$$

*where the prefactor $Y_L$ is*

$$
\begin{aligned}
&Y_L(\{\mu_{\sigma(i)}\}, \{\mu^{(1)}_{\sigma(j)}\}) \\
&= \frac{1}{p!(n-p)!} c^{\sigma}_{1 \ldots n} B^{**}_{n-p}\left(\mu^{(1)}_{p+1}, \ldots, \mu^{(1)}_n | \mu_{\sigma(p+1)}, \ldots, \mu_{\sigma(n)}\right) \prod_{k=p+1}^{n} \prod_{l=1}^{p} \left(-\frac{2a(\mu_{\sigma(k)}, \mu_{\sigma(l)})}{a(\mu_{\sigma(l)}, \mu_{\sigma(k)})}\right),
\end{aligned}
$$
$$(3.24)$$

*and the c-number $B^{**}_p$ is given by*

$$
\begin{aligned}
&B^{**}_p\left(\mu^{(1)}_1, \ldots, \mu^{(1)}_p | \mu_1, \ldots, \mu_p\right) \\
&= \sum_{\sigma \in \mathcal{S}_p} \prod_{k=1}^{p} b(\mu^{(1)}_k, \mu_{\sigma(k)}) \prod_{j \neq \sigma(k), \ldots, \sigma(p)} c(\mu^{(1)}_k, \mu_j) \prod_{l=k+1}^{p} \frac{a(\mu^{(1)}_k, \mu_{\sigma(l)})}{2a(\mu_{\sigma(l)}, \mu_{\sigma(k)})}.
\end{aligned}
$$

# 4 Determinant representation of the scalar product

The scalar product of the Bethe states with a given quantum number $p$ is defined by

$$
P_n(\{\mu_i\}_{(p,n)}, \{\lambda_j\}_{(p,n)}) = \langle \Omega_N(\{\mu_j\}_{(p,n)}) | \Omega_N(\{\lambda_j\}_{(p,n)}) \rangle. \qquad (4.1)
$$

11

The invariant property of the pseudo-vacuum state under the $F$-transformation, i.e. $F_{1...N}|0\rangle = |0\rangle$ and $\langle 0|F_{1...N}^{-1} = \langle 0|$, implies that in the $F$-basis, the scalar product $P_n$ is

$$
\begin{aligned}
P_n(\{\mu_i\}_{(p,n)}, \{\lambda_j\}_{(p,n)}) &= \langle \tilde{\Omega}_N(\{\mu_j\}_{(p,n)})|\tilde{\Omega}_N(\{\lambda_j\}_{(p,n)})\rangle \\
&= \sum_{\sigma',\sigma} Y_L(\{\mu_{\sigma'(j)}\}, \{\mu^{(1)}_{\sigma'(k)}\}) Y_R(\{\lambda_{\sigma(j)}\}, \{\lambda^{(1)}_{\sigma(k)}\}) \\
&\quad \times \langle 0|\tilde{B}_1(\mu_{\sigma'(n)})\ldots\tilde{B}_1(\mu_{\sigma'(p+1)})\tilde{B}_2(\mu_{\sigma'(p)})\ldots\tilde{B}_2(\mu_{\sigma'(1)}) \\
&\quad \times \tilde{C}_2(\lambda_{\sigma(1)})\ldots\tilde{C}_2(\lambda_{\sigma(p)})\tilde{C}_1(\lambda_{\sigma(p+1)})\ldots\tilde{C}_1(\lambda_{\sigma(n)})|0\rangle. \quad (4.2)
\end{aligned}
$$

To compute the scalar product, following [12], we introduce the following intermediate functions

$$
\begin{aligned}
&G^{(m)}(\{\lambda_k\}_{(p,n)}, u_1, \ldots, \mu_m, i_{m+1}, \ldots, i_n) \\
&= \begin{cases}
\langle 0|\overset{\overleftarrow{n}}{\prod_{k=p+1}} E^{31}_{(i_k)} \overset{\overleftarrow{p}}{\prod_{k=m+1}} E^{32}_{(i_k)} \tilde{B}_2(\mu_m)\ldots\tilde{B}_2(\mu_1) \\
\quad \times \tilde{C}_2(\lambda_1)\ldots\tilde{C}_2(\lambda_p)\tilde{C}_1(\lambda_{p+1})\ldots\tilde{C}_1(\lambda_n)|0> \quad \text{for } m \leq p, \\[2ex]
\langle 0|\overset{\overleftarrow{n}}{\prod_{k=m+1}} E^{31}_{(i_k)} \tilde{B}_1(\mu_m)\ldots\tilde{B}_1(\mu_{p+1})\tilde{B}_2(\mu_p)\tilde{B}_2(\mu_1) \\
\quad \times \tilde{C}_2(\lambda_1)\ldots\tilde{C}_2(\lambda_p)\tilde{C}_1(\lambda_{p+1})\ldots\tilde{C}_1(\lambda_n)|0> \quad \text{for } m \geq p+1.
\end{cases} \quad (4.3)
\end{aligned}
$$

where the lower indices of $E^{32}_{(i_k)}$ and $E^{31}_{(i_k)}$ satisfy the relations $i_{m+1} < \ldots < i_p$, $i_{p+1} < \ldots < i_n$ and $\{i_1, \ldots, i_p\} \cap \{i_{p+1}, \ldots, i_n\} = \varnothing$. Thus, the scalar product can be rewritten as

$$
\begin{aligned}
&P_n(\{\mu_i\}_{(p,n)}, \{\lambda_j\}_{(p,n)}) \\
&= \sum_{\sigma,\sigma'} Y_L(\{\mu_{\sigma'(j)}\}, \{\mu^{(1)}_{\sigma'(k)}\}) Y_R(\{\lambda_{\sigma(j)}\}, \{\lambda^{(1)}_{\sigma(k)}\}) \; G^{(n)}(\{\lambda_{\sigma(j)}\}_{(p,n)}, \{\mu_{\sigma'(k)}\}_{(p,n)}). \quad (4.4)
\end{aligned}
$$

We now compute $G^{(m)}$ for $m \leq p$ and $m \geq p+1$ separately.

## 4.1 $\quad 1 \leq m \leq p$

We first compute the function $G^{(m)}$ for $m \leq p$.

Inserting a complete set, (4.3) becomes

$$
\begin{aligned}
&G^{(m)}(\{\lambda_k\}_{(p,n)}, \mu_1, \ldots, \mu_m, i_{m+1}, \ldots, i_n) \\
&= \sum_{j \neq i_{m+1}, \ldots, i_n}^{N} \langle 0|\overset{\overleftarrow{n}}{\prod_{k=p+1}} E^{31}_{(i_k)} \overset{\overleftarrow{p}}{\prod_{k=m+1}} E^{32}_{(i_k)} \tilde{B}_2(\mu_m) \overset{\overrightarrow{m+q}}{\prod_{k=m+1}} E^{23}_{(i_k)} E^{23}_{(j)} \overset{\overrightarrow{p}}{\prod_{k=m+q+1}} E^{23}_{(i_k)} \overset{\overrightarrow{n}}{\prod_{k=p+1}} E^{13}_{(i_k)}|0\rangle \\
&\quad \times G^{(m-1)}(\{\lambda_k\}_{(p,n)}, \mu_1, \ldots, \mu_{m-1}, i_{m+1}, \ldots, i_{m+q}, j, i_{m+q+1} \ldots, i_n) \quad (0 \leq q \leq p - m).
\end{aligned}
$$

$$(4.5)$$

12

In view of (3.11), we have

$$\langle 0| \overset{\overleftarrow{n}}{\underset{k=p+1}{\prod}} E_{(i_k)}^{31} \overset{\overleftarrow{p}}{\underset{k=m+1}{\prod}} E_{(i_k)}^{32} \tilde{B}_2(\mu_m) \overset{\overrightarrow{m+q}}{\underset{k=m+1}{\prod}} E_{(i_k)}^{23} E_{(j)}^{23} \overset{\overrightarrow{p}}{\underset{k=m+q+1}{\prod}} E_{(i_k)}^{23} \overset{\overrightarrow{n}}{\underset{k=p+1}{\prod}} E_{(i_k)}^{13} |0\rangle$$

$$= -(-1)^q 2^{-(p-m)} \cdot b(\mu_m, \xi_j) \prod_{l=m+1}^{p} a(\mu_m, \xi_{i_l}) \prod_{l=p+1}^{n} a(\mu_m, \xi_{i_l}) \prod_{k \neq j, i_{p+1}, \dots, i_n}^{N} a^{-1}(\xi_k, \xi_j).$$

$$(4.6)$$

Substituting the expressions of $\tilde{C}_1$ (3.10) and $\tilde{C}_2$ (3.9) into (4.3), we obtain $G^{(0)}$:

$$G^{(0)}(\{\lambda_k\}_{(p,n)}, i_1, \dots, i_n) = \langle 0| \overset{\overleftarrow{n}}{\underset{k=p+1}{\prod}} E_{(i_k)}^{31} \overset{\overleftarrow{p}}{\underset{k=1}{\prod}} E_{(i_k)}^{32} \overset{p}{\underset{k=1}{\prod}} \tilde{C}_2(\lambda_k) \overset{n}{\underset{k=p+1}{\prod}} \tilde{C}_1(\lambda_k) |0 >$$

$$= 2^{\frac{p(p-1)+(n-p)(n-p-1)}{2}} \prod_{l=1}^{p} \prod_{k=p+1}^{n} a(\lambda_l, \xi_{i_k}) \det\mathcal{B}_p(\lambda_1, \dots, \lambda_p; \xi_{i_1}, \dots, \xi_{i_p})$$

$$\times \det\mathcal{B}_{n-p}(\lambda_{p+1}, \dots, \lambda_n; \xi_{i_{p+1}}, \dots, \xi_{i_n}). \qquad (4.7)$$

We compute $G^{(m)}$ by using the recursion relation (4.5). One sees that there are two different determinants in $G^{(0)}$, which are labelled by different $\lambda$'s and $\xi_{i_k}$'s. For $m \leq p$ we only focus on the first determinant, i.e. $\det\mathcal{B}_p$.

To compute $G^{(1)}$, we substitute (4.6) and (4.7) into (4.5) to obtain

$$G^{(1)}(\{\lambda_k\}_{(p,n)}, \mu_1, i_2, \dots, i_n)$$

$$= \sum_{j \neq i_2, \dots, i_n}^{N} \langle 0| \overset{\overleftarrow{n}}{\underset{k=p+1}{\prod}} E_{(i_k)}^{31} \overset{\overleftarrow{p}}{\underset{k=2}{\prod}} E_{(i_k)}^{32} \tilde{B}_2(\mu_1) \overset{\overrightarrow{q+1}}{\underset{k=2}{\prod}} E_{(i_k)}^{23} E_{(j)}^{23} \overset{\overrightarrow{p}}{\underset{q+2}{\prod}} E_{(i_k)}^{23} \overset{\overrightarrow{n}}{\underset{k=p+1}{\prod}} E_{(i_k)}^{13} |0\rangle$$

$$\times G^{(0)}(\{\lambda_k\}_{(p,n)}, i_2, \dots, i_{q+1}, j, i_{q+2}, \dots, i_n)$$

$$= -2^{\frac{(p-1)(p-2)+(n-p)(n-p-1)}{2}} \prod_{l=1}^{p} \prod_{k=p+1}^{n} a(\lambda_l, \xi_{i_k}) \sum_{j \neq i_2, \dots, i_n}^{N} (-1)^q b(\mu_1, \xi_j)$$

$$\times \prod_{l=2}^{p} a(\mu_1, \xi_{i_l}) \prod_{k \neq j, i_{p+1}, \dots, i_n}^{N} a^{-1}(\xi_k, \xi_j) \det\mathcal{B}_p(\lambda_1 \dots, \lambda_p; \xi_{i_2}, \dots, \xi_{i_{q+1}}, \xi_j, \xi_{i_{q+2}}, \dots, \xi_{i_p})$$

$$\times \prod_{l=p+1}^{n} a(\mu_1, \xi_{i_l}) \det\mathcal{B}_{n-p}(\lambda_{p+1}, \dots, \lambda_n; \xi_{i_{p+1}}, \dots, \xi_{i_n}). \qquad (4.8)$$

Let $\lambda_k$ $(k = 1, \dots, n)$ label the row and $\xi_l$ $(l = i_2, \dots, j, \dots, i_p)$ label the column of the matrix $\mathcal{B}_p$. From (4.7), one sees that the column indices in (4.8) satisfy the sequence $i_2 < \dots < j <$

$\ldots < i_p$. Therefore, moving the $j$th column in the matrix $\mathcal{B}_p$ to the first column, we have

$$
\begin{aligned}
&G^{(1)}(\{\lambda_k\}_{(p,n)}, \mu_1, i_2, \ldots, i_n) \\
&= -2^{\frac{(p-1)(p-2)+(n-p)(n-p-1)}{2}} \prod_{l=1}^{p} \prod_{k=p+1}^{n} a(\lambda_l, \xi_{i_k}) \sum_{j \neq i_2, \ldots, i_n} b(\mu_1, \xi_j) \\
&\quad \times \prod_{l=2}^{p} a(\mu_1, \xi_{i_l}) \prod_{k \neq j, i_{p+1}, \ldots, i_n}^{N} a^{-1}(\xi_k, \xi_j) \det \mathcal{B}_p(\lambda_1, \ldots, \lambda_p; \xi_j, \xi_{i_2}, \ldots, \xi_{i_n}) \\
&\quad \times \det \mathcal{B}_{n-p}(\lambda_{p+1}, \ldots, \lambda_n; \xi_{i_{p+1}}, \ldots, \xi_{i_n}) \\
&= -2^{\frac{(p-1)(p-2)+(n-p)(n-p-1)}{2}} \det \mathcal{B}_p^{(1)}(\lambda_1, \ldots, \lambda_p; \mu_1, \xi_{i_2}, \ldots, \xi_{i_p}) \\
&\quad \times \prod_{l=p+1}^{n} a(\mu_1, \xi_{i_l}) \det \mathcal{B}_{n-p}(\lambda_{p+1}, \ldots, \lambda_n; \xi_{i_{p+1}}, \ldots, \xi_{i_n}), \quad (4.9)
\end{aligned}
$$

where the matrix $\tilde{\mathcal{B}}_p^{(1)}(\{\lambda_k\}, \mu_1, \xi_{i_2}, \ldots, \xi_{i_p})$ is given by

$$
(\mathcal{B}_p^{(1)})_{\alpha\beta} = \prod_{k=p+1}^{n} a(\lambda_\alpha, \xi_{i_k}) \, a(\mu_1, \xi_{i_\beta})(\mathcal{B}_p)_{\alpha\beta} \qquad (1 \leq \alpha \leq p \text{ and } 2 \leq \beta \leq p), \quad (4.10)
$$

$$
\begin{aligned}
(\mathcal{B}_p^{(1)})_{\alpha 1} &= \prod_{k=p+1}^{n} a(\lambda_\alpha, \xi_{i_k}) \sum_{j \neq i_2, \ldots, i_n} b(\mu_1, \xi_j) b(\lambda_\alpha, \xi_j) \prod_{\gamma=1}^{\alpha-1} a(\lambda_\gamma, \xi_j) \\
&\quad \times \prod_{k \neq j, i_{p+1}, \ldots, i_n}^{N} a^{-1}(\xi_k, \xi_j) \qquad (1 \leq \alpha \leq p). \quad (4.11)
\end{aligned}
$$

Using the properties of determinant, one finds that if $j = i_2, \ldots, i_p$, the corresponding terms in (4.11) contribute zero to the determinant. Thus, without changing the determinant of the matrix $\mathcal{B}_p^{(1)}$, the elements $(\mathcal{B}_p^{(1)})_{\alpha 1}$ in (4.11) may be replaced by

$$
\begin{aligned}
(\mathcal{B}_p^{(1)})_{\alpha 1} &= \prod_{k=p+1}^{n} \frac{\lambda_\alpha - \xi_{i_k}}{\lambda_\alpha - \xi_{i_k} + \eta} \sum_{j \neq i_{p+1}, \ldots, i_n} \frac{\eta}{\mu_1 - \xi_j + \eta} \frac{\eta}{\lambda_\alpha - \xi_j + \eta} \\
&\quad \times \prod_{\gamma=1}^{\alpha-1} \frac{\lambda_\gamma - \xi_j}{\lambda_\gamma - \xi_j + \eta} \prod_{k \neq j, i_{p+1}, \ldots, i_n}^{N} \frac{\xi_k - \xi_j + \eta}{\xi_k - \xi_j}. \quad (4.12)
\end{aligned}
$$

Thanks to the Bethe ansatz equation (2.22), we may construct the function

$$
\begin{aligned}
\mathcal{M}_{\alpha\beta} &= \frac{\eta}{\lambda_\alpha - \mu_\beta} \prod_{\gamma=1}^{\alpha-1} \frac{\lambda_\gamma - \mu_\beta - \eta}{\lambda_\gamma - \mu_\beta} \prod_{\epsilon=1}^{\beta-1} \frac{\mu_\epsilon - \mu_\beta - \eta}{\mu_\epsilon - \mu_\beta} \left[ \prod_{j=p+1}^{n} \frac{\mu_\beta - \lambda_j^{(1)}}{\mu_\beta - \lambda_j^{(1)} + \eta} \right. \\
&\quad \left. - \prod_{k=1}^{N} \frac{\mu_\beta - \xi_k}{\mu_\beta - \xi_k + \eta} \prod_{j=p+1}^{n} \frac{\mu_\beta - \xi_{i_j} + \eta}{\mu_\beta - \xi_{i_j}} \frac{\lambda_\alpha - \xi_{i_j}}{\lambda_\alpha - \xi_{i_j} + \eta} \right]
\end{aligned}
$$

14

$$+ \sum_{j=p+1}^{n} \left[ \frac{\eta}{\mu_\beta - \lambda_j^{(1)} + \eta} \frac{\eta}{\lambda_\alpha - \lambda_j^{(1)} + \eta} \prod_{\gamma=1}^{\alpha-1} \frac{\lambda_\gamma - \lambda_j^{(1)}}{\lambda_\gamma - \lambda_j^{(1)} + \eta} \right.$$

$$\left. \times \prod_{\epsilon=1}^{\beta-1} \frac{\mu_\epsilon - \lambda_j^{(1)}}{\mu_\epsilon - \lambda_j^{(1)} + \eta} \prod_{k=p+1,\neq j}^{n} \frac{\lambda_k^{(1)} - \lambda_j^{(1)} + \eta}{\lambda_k^{(1)} - \lambda_j^{(1)}} \right]$$

$$+ \sum_{\gamma=1}^{\alpha-1} \frac{\eta}{\lambda_\alpha - \lambda_\gamma} \frac{\eta}{\lambda_\gamma - \mu_\beta} \prod_{\iota=1,\neq\gamma}^{\alpha-1} \frac{\lambda_\iota - \lambda_\gamma - \eta}{\lambda_\iota - \lambda_\gamma} \prod_{\epsilon=1}^{\beta-1} \frac{\mu_\epsilon - \lambda_\gamma - \eta}{\mu_\epsilon - \lambda_\gamma} \left[ \prod_{j=p+1}^{n} \frac{\lambda_\gamma - \lambda_j^{(1)}}{\lambda_\gamma - \lambda_j^{(1)} + \eta} \right.$$

$$\left. - \prod_{k=1}^{N} \frac{\lambda_\gamma - \xi_k}{\lambda_\gamma - \xi_k + \eta} \prod_{j=p+1}^{n} \frac{\lambda_\gamma - \xi_{i_j} + \eta}{\lambda_\gamma - \xi_{i_j}} \frac{\lambda_\alpha - \xi_{i_j}}{\lambda_\alpha - \xi_{i_j} + \eta} \right]$$

$$+ \sum_{\epsilon=1}^{\beta-1} \frac{\eta}{\lambda_\alpha - \mu_\epsilon} \frac{\eta}{\mu_\epsilon - \mu_\beta} \prod_{\gamma=1}^{\alpha-1} \frac{\lambda_\gamma - \mu_\epsilon - \eta}{\lambda_\gamma - \mu_\epsilon} \prod_{\varepsilon=1,\neq\epsilon}^{\beta-1} \frac{\mu_\varepsilon - \mu_\epsilon - \eta}{\mu_\varepsilon - \mu_\epsilon} \left[ \prod_{j=p+1}^{n} \frac{\mu_\epsilon - \lambda_j^{(1)}}{\mu_\epsilon - \lambda_j^{(1)} + \eta} \right.$$

$$\left. - \prod_{k=1}^{N} \frac{\mu_\epsilon - \xi_k}{\mu_\epsilon - \xi_k + \eta} \prod_{j=p+1}^{n} \frac{\mu_\epsilon - \xi_{i_j} + \eta}{\mu_\epsilon - \xi_{i_j}} \frac{\lambda_\alpha - \xi_{i_j}}{\lambda_\alpha - \xi_{i_j} + \eta} \right], \tag{4.13}$$

where $\lambda_j^{(1)}$ $(j = p+1, \ldots, n)$ satisfy the NBAE (2.23). By direct computation, one sees that the residues of $\mathcal{M}_{\alpha 1}$ at points $\mu_1 = \lambda_j^{(1)} - \eta$, $\mu_1 = \lambda_\gamma$ $(\gamma = 1, \ldots, \alpha - 1)$ and $\mu_1 = \mu_\epsilon$ $(\epsilon = 1, \ldots, \beta - 1)$ are zero. Moreover the residues of $\mathcal{M}_{\alpha 1}$ at the points $\mu_1 = \lambda_\alpha$ are also zero because $\lambda_\alpha$ is a solution of the BAE (2.22). Then comparing (4.12) with (4.13), one finds that as functions of $\mu_1$, the functions $(\mathcal{B}_p^{(1)})_{\alpha 1}$ and $\mathcal{M}_{\alpha 1}$ have the same residues at the simple poles $\mu_1 = \xi_j - \eta$ $(j \neq i_{p+1}, \ldots, i_n)$, and that when $\mu_1 \to \infty$, they tend to zero. Therefore, according to the properties of the rational functions, we have $(\mathcal{B}_p^{(1)})_{\alpha 1} = \mathcal{M}_{\alpha 1}$.

Then, by using the function $G^{(0)}, G^{(1)}$ and the intermediate function (4.5) repeatedly, we obtain $G^{(m)}$ $(m \leq p)$:

$$G^{(m)}(\{\lambda_k\}_{(p,n)}, \mu_1, \ldots, \mu_m, i_{m+1}, \ldots, i_n)$$
$$= (-1)^m 2^{\frac{p(p-1)-m(2p-m-1)+(n-p)(n-p-1)}{2}} \det \mathcal{B}_p^{(m)}(\lambda_1, \ldots, \lambda_p; \mu_1, \ldots, \mu_m, i_{m+1}, \ldots, i_p)$$
$$\times \prod_{l=1}^{m} \prod_{k=p+1}^{n} a(\mu_l, \xi_{i_k}) \det \mathcal{B}_{n-p}(\lambda_{p+1}, \ldots, \lambda_n; \xi_{i_{p+1}}, \ldots, \xi_{i_n}) \tag{4.14}$$

with the matrix elements

$$(\mathcal{B}_p^{(m)})_{\alpha\beta} = \prod_{\epsilon=1}^{\beta-1} a(\mu_\epsilon, \xi_{i_\beta})(\mathcal{B}_p)_{\alpha\beta}, \qquad (1 \leq \alpha \leq p, \ m < \beta \leq p),$$

$$(\mathcal{B}_p^{(m)})_{\alpha\beta} = \mathcal{M}_{\alpha\beta}, \qquad (1 \leq \alpha \leq p, \ 1 \leq \beta \leq m). \tag{4.15}$$

(4.14) can be proved by induction. Firstly from (4.9), (4.10) and (4.13), (4.14) is true for $m = 1$. Assume (4.14) for $G^{(m-1)}$. Let us show (4.14) for general $m$. Substituting $G^{(m-1)}$

and (4.6) into intermediate function (4.5), we have

$$
\begin{aligned}
& G^{(m)}(\{\lambda_k\}_{(p,n)}, \mu_1, \ldots, \mu_m, i_{m+1}, \ldots, i_n) \\
& = -(-1)^q 2^{-(p-m)} \sum_{j \neq i_{m+1}, \ldots, i_n} b(\mu_m, \xi_j) \prod_{l=m+1}^{n} a(\mu_m, \xi_{i_l}) \prod_{k \neq j, p+1, \ldots, n}^{N} a^{-1}(\xi_k, \xi_j) \\
& \quad \times G^{(m-1)}(\{\lambda_k\}_{(p,n)}, \mu_1, \ldots, \mu_{m-1}, i_{m+1}, \ldots, i_{m+p}, j, i_{m+p+1}, \ldots, i_n) \\
& = -2^{-(p-m)} \sum_{j \neq i_{m+1}, \ldots, i_n} b(\mu_m, \xi_j) \prod_{l=m+1}^{n} a(\mu_m, \xi_{i_l}) \prod_{k \neq j, p+1, \ldots, n}^{N} a^{-1}(\xi_k, \xi_j) \\
& \quad \times G^{(m-1)}(\{\lambda_k\}_{(p,n)}, \mu_1, \ldots, \mu_{m-1}, j, i_{m+1}, \ldots, i_n) \\
& = (-1)^m 2^{\frac{p(p-1)-m(2p-m-1)+(n-p)(n-p+1)}{2}} \det\mathcal{B}_p^{(m)}(\lambda_1, \ldots, \lambda_p; \mu_1 \ldots, \mu_m, i_{m+1}, \ldots, i_p) \\
& \quad \times \prod_{l=1}^{m} \prod_{k=p+1}^{n} a(\mu_l, \xi_{i_k}) \det\mathcal{B}_{n-p}(\lambda_{p+1}, \ldots, \lambda_n; \xi_{i_{p+1}}, \ldots, \xi_{i_n}),
\end{aligned}
\tag{4.16}
$$

where the matrix elements $(\mathcal{B}_p^{(m)})_{\alpha\beta}$ are given by

$$
\begin{aligned}
(\mathcal{B}_p^{(m)})_{\alpha\beta} &= \prod_{\epsilon=1}^{\beta-1} a(\mu_\epsilon, \xi_{i_\beta}) \prod_{k=p+1}^{n} a(\lambda_\alpha, \xi_{i_k})(\mathcal{B}_p)_{\alpha\beta} && (1 \leq \alpha \leq p, \ m < \beta \leq p), \\
(\mathcal{B}_p^{(m)})_{\alpha\beta} &= \mathcal{M}_{\alpha\beta} && (1 \leq \alpha \leq p, \ 1 \leq \beta < m), \\
(\mathcal{B}_p^{(m)})_{\alpha m} &= \prod_{k=p+1}^{n} a(\lambda_\alpha, \xi_{i_k}) \sum_{j \neq i_{m+1}, \ldots, i_n} b(\mu_m, \xi_j) b(\lambda_\alpha, \xi_j) \prod_{\gamma=1}^{\alpha-1} a(\lambda_\gamma, \xi_j) \\
& \quad \times \prod_{\epsilon=1}^{\beta-1} a(\mu_\epsilon, \xi_j) \prod_{k \neq j, i_{p+1}, \ldots, i_n}^{N} a^{-1}(\xi_k, \xi_j) && (1 \leq \alpha \leq p).
\end{aligned}
\tag{4.17}
$$

By the procedure leading to $(\mathcal{B}_p^{(1)})_{\alpha\beta}$, we prove $(\mathcal{B}_p^{(m)})_{\alpha m} = \mathcal{M}_{\alpha m}$. Therefore we have proved that the function (4.14) holds for all $m \leq p$.

When $m = p$, we have,

$$
\begin{aligned}
& G^{(p)}(\{\lambda_k\}_{(p,n)}, \mu_1, \ldots, \mu_p, i_{p+1}, \ldots, i_n) \\
& = (-1)^p 2^{\frac{(n-p)(n-p-1)}{2}} \det\mathcal{M}(\lambda_1, \ldots, \lambda_p; \mu_1, \ldots, \mu_p) \\
& \quad \times \prod_{l=1}^{p} \prod_{k=p+1}^{n} a(\mu_l, \xi_{i_k}) \det\mathcal{B}_{n-p}(\lambda_{p+1}, \ldots, \lambda_n; \xi_{i_{p+1}}, \ldots, \xi_{i_n}),
\end{aligned}
\tag{4.18}
$$

where the matrix elements of $\mathcal{M}$ are given by (4.13).

For later use, we rewrite the element of the matrix $\mathcal{M}_{\alpha\beta}$ $(1 \leq \alpha, \beta \leq p)$ in the form

$$
\mathcal{M}_{\alpha\beta} = \mathcal{F}_{\alpha\beta} + \sum_{\epsilon=1}^{\beta} \prod_{j=p+1}^{n} \left( a^{-1}(\mu_\epsilon, \xi_{i_j}) a(\lambda_\alpha, \xi_{i_j}) \right) \mathcal{G}_{\alpha\beta}^\epsilon
$$

16

$$+ \sum_{\gamma=1}^{\alpha-1} \prod_{j=p+1}^{n} \left( a^{-1}(\lambda_\gamma, \xi_{i_j}) a(\lambda_\alpha, \xi_{i_j}) \right) \mathcal{H}_{\alpha\beta}^\gamma, \tag{4.19}$$

where

$$
\begin{aligned}
\mathcal{F}_{\alpha\beta} &= \frac{b(\lambda_\alpha, \mu_\beta)}{a(\lambda_\alpha, \mu_\beta)} \prod_{\gamma=1}^{\alpha-1} a^{-1}(\mu_\beta, \lambda_\gamma) \prod_{\epsilon=1}^{\beta-1} a^{-1}(\mu_\beta, \mu_\epsilon) \prod_{j=p+1}^{n} a(\mu_\beta, \lambda_j^{(1)}) \\
&\quad + \sum_{j=p+1}^{n} \left[ b(\mu_\beta, \lambda_j^{(1)}) b(\lambda_\alpha, \lambda_j^{(1)}) \prod_{\gamma=1}^{\alpha-1} a(\lambda_\gamma, \lambda_j^{(1)}) \prod_{\epsilon=1}^{\beta-1} a(\mu_\epsilon, \lambda_j^{(1)}) \prod_{k \neq j} a^{-1}(\lambda_k^{(1)}, \lambda_j^{(1)}) \right] \\
&\quad + \sum_{\gamma=1}^{\alpha-1} \frac{b(\lambda_\alpha, \lambda_\gamma)}{a(\lambda_\alpha, \lambda_\gamma)} \frac{b(\lambda_\gamma, \mu_\beta)}{a(\lambda_\gamma, \mu_\beta)} \prod_{\iota=1, \neq\gamma}^{\alpha-1} a^{-1}(\lambda_\gamma, \lambda_\iota) \prod_{\epsilon=1}^{\beta-1} a^{-1}(\lambda_\gamma, \mu_\epsilon) \prod_{j=p+1}^{n} a(\lambda_\gamma, \lambda_j^{(1)}) \\
&\quad + \sum_{\epsilon=1}^{\beta-1} \frac{b(\lambda_\alpha, \mu_\epsilon)}{a(\lambda_\alpha, \mu_\epsilon)} \frac{b(\mu_\epsilon, \mu_\beta)}{a(\mu_\epsilon, \mu_\beta)} \prod_{\gamma=1}^{\alpha-1} a^{-1}(\mu_\epsilon, \lambda_\gamma) \prod_{\varepsilon=1, \neq\epsilon}^{\beta-1} a^{-1}(\mu_\epsilon, \mu_\varepsilon) \prod_{j=p+1}^{n} a(\mu_\epsilon, \lambda_j^{(1)}),
\end{aligned}
\tag{4.20}
$$

$$
\mathcal{G}_{\alpha\beta}^\epsilon = 
\begin{cases}
-\dfrac{b(\lambda_\alpha, \mu_\epsilon)}{a(\lambda_\alpha, \mu_\epsilon)} \displaystyle\prod_{\gamma=1}^{\alpha-1} a^{-1}(\mu_\epsilon, \lambda_\gamma) \prod_{k=1}^{N} a(\mu_\beta, \xi_k) & (\epsilon = \beta) \\[2.5ex]
-\dfrac{b(\lambda_\alpha, \mu_\epsilon)}{a(\lambda_\alpha, \mu_\epsilon)} \dfrac{b(\mu_\epsilon, \mu_\beta)}{a(\mu_\epsilon, \mu_\beta)} \displaystyle\prod_{\gamma=1}^{\alpha-1} a^{-1}(\mu_\epsilon, \lambda_\gamma) \prod_{\varepsilon=1, \neq\epsilon}^{\beta-1} a^{-1}(\mu_\epsilon, \mu_\varepsilon) \prod_{k=1}^{N} a(\mu_\epsilon, \xi_k) & (1 \leq \epsilon \leq \beta-1)
\end{cases}
\tag{4.21}
$$

$$
\mathcal{H}_{\alpha\beta}^\gamma = -\frac{b(\lambda_\alpha, \lambda_\gamma)}{a(\lambda_\alpha, \lambda_\gamma)} \frac{b(\lambda_\gamma, \mu_\beta)}{a(\lambda_\gamma, \mu_\beta)} \prod_{\iota=1, \neq\gamma}^{\alpha-1} a^{-1}(\lambda_\gamma, \lambda_\iota) \prod_{\epsilon=1}^{\beta-1} a^{-1}(\lambda_\gamma, \mu_\epsilon) \prod_{k=1}^{N} a(\lambda_\gamma, \xi_k). \tag{4.22}
$$

After a tedious computation, we obtain the determinant of the matrix $\mathcal{M}$

$$
\begin{aligned}
&\det \mathcal{M}(\{\lambda_\alpha\}, \{\mu_\beta\}) \\
&= \det \mathcal{F}(\{\lambda_\alpha\}, \{\mu_\beta\}) \\
&\quad + \sum_{k, l_k, \varrho_{m_k}^k}{}' \prod_{e=1}^{k} \prod_{g=p+1}^{n} a^{-l_e}(\mu_e, \xi_{i_g}) \sum_{\substack{j_1, \ldots, j_p=1 \\ j_1 \neq \ldots \neq j_p}}^{p} (-1)^{\tau(j_1 j_2 \ldots j_p)} \\
&\quad \times \prod_{f'=1}^{p} \prod_{f=p+1}^{n} \prod_{t=1}^{k} \prod_{t'=1}^{l_t} \left[ 1 + \delta_{j_{f'} \varrho_{t'}^t} \left( a(\lambda_{f'}, \xi_{i_f}) - 1 \right) \right] \\
&\quad \times (A_2)_{1\, j_1} (A_2)_{2\, j_2} \ldots (A_2)_{p\, j_p} \\
&\quad + \sum_{k, l_k, \rho_{m_k}^k}{}'' \prod_{e=1}^{k} \prod_{g=p+1}^{n} a^{-l_e}(\lambda_e, \xi_{i_g}) \prod_{t=1}^{k} \prod_{t'=1}^{l_t} \prod_{g'=p+1}^{n} a(\lambda_{\rho_{t'}^t}, \xi_{i_{g'}}) \det A_3(\{\lambda_\alpha\}, \{\mu_\beta\})
\end{aligned}
$$

$$+ \sum_{k,l_k,\varrho^k_{m_k}}{}' \sum_{k',l'_{k'},\rho^{k'}_{m'_{k'}}}{}'' \prod_{e=1}^{k} \prod_{g=p+1}^{n} a^{-l'_e}(\lambda_e, \xi_{i_g}) \prod_{t=1}^{k'} \prod_{t'=1}^{l'_t} \prod_{g'=p+1}^{n} \left( a(\lambda_{\rho^t_{t'}}, \xi_{i_{g'}}) \right)$$

$$\times \sum_{\substack{j_1,\ldots,j_p=1 \\ j_1 \neq \ldots \neq j_p}}^{p} (-1)^{\tau(j_1 j_2 \ldots j_p)}$$

$$\times \prod_{t=1}^{k} \prod_{f=p+1}^{n} \prod_{\substack{s=1 \\ s \neq \{\rho^1_{m_1}\},\ldots,\{\rho^k_{m_k}\}}}^{p} \prod_{f'=1}^{p} \prod_{t'=1}^{l_t} \left[ 1 + \delta_{f' s}\, \delta_{j_{f'}\, \varrho^t_{t'}}\, \left( a^{-1}(\mu_t, \xi_{i_f}) - 1 \right) \right]$$

$$\times \prod_{t=1}^{k} \prod_{f=p+1}^{n} \prod_{\substack{s=1 \\ s \neq \{\rho^1_{m_1}\},\ldots,\{\rho^k_{m_k}\}}}^{p} \prod_{f'=1}^{p} \prod_{t'=1}^{l_t} \left[ 1 + \delta_{f' s}\, \delta_{j_{f'}\, \varrho^t_{t'}}\, \left( a(\lambda_s, \xi_{i_f}) - 1 \right) \right]$$

$$\times (A_4)_{1\,j_1}(A_4)_{2\,j_2} \ldots (A_4)_{p\,j_p} \tag{4.23}$$

$$\equiv \quad \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4, \tag{4.24}$$

where $\tau(x_1,\ldots,x_p) = \tau(\sigma)$, $(\sigma \in \mathcal{S}_p$ and $(x_1,\ldots,x_p) = \sigma(1,\ldots,p))$, $\tau(\sigma) = 0$ if $\sigma$ is even and $\tau(\sigma) = 1$ if $\sigma$ is odd,

$$\sum_{k,l_k,\varrho^k_{m_k}}{}' = \sum_{k=1}^{p} \left\{ \sum_{l_k=1}^{p-k+1} \frac{1}{l_k!} \prod_{m_k=1}^{l_k} \sum_{\substack{\varrho^k_{m_k}=k \\ \varrho^k_{m_k} \neq \varrho^k_{m_k-1},\ldots,\varrho^k_1}}^{p} \right\} \left\{ \prod_{r=1}^{k-1} \sum_{l_{k-r}=0}^{p-k+r+1-\sum_{j=0}^{r-1} l_{k-j}} \frac{1}{l_{k-r}!} \right.$$

$$\left. \times \prod_{m_{k-r}=1}^{l_{k-r}} \sum_{\substack{\varrho^{k-r}_{m_{k-r}}=k-r \\ \varrho^{k-r}_{m_{k-r}} \neq \varrho^{k-r}_{m_{k-r}-1},\ldots,\varrho^{k-r}_1 \\ \varrho^{k-r}_{m_{k-r}} \neq \{\varrho^{k-r+1}_\kappa\},\{\varrho^{k-r+2}_\kappa\},\ldots,\{\varrho^k_\kappa\}}}^{p} \right\},$$

$$\sum_{k,l_k,\rho^k_{m_k}}{}'' = \sum_{k=1}^{p-1} \left\{ \sum_{l_k=1}^{p-k} \frac{1}{l_k!} \prod_{m_k=1}^{l_k} \sum_{\substack{\rho^k_{m_k}=k+1 \\ \rho^k_{m_k} \neq \rho^k_{m_k-1},\ldots,\rho^k_1}}^{p} \right\} \left\{ \prod_{r=1}^{k-1} \sum_{l_{k-r}=0}^{p-k+r-\sum_{j=0}^{r-1} l_{k-j}} \frac{1}{l_{k-r}!} \right.$$

$$\times \prod_{m_{k-r}=1}^{l_{k-r}} \sum_{\substack{\rho^{k-r}_{m_{k-r}}=k-r+1 \\ \rho^{k-r}_{m_{k-r}} \neq \rho^{k-r}_{m_{k-r}-1},\ldots,\rho^{k-r}_1 \\ \rho^{k-r}_{m_{k-r}} \neq \{\rho^{k-r+1}_\kappa\},\{\rho^{k-r+2}_\kappa\},\ldots,\{\rho^k_\kappa\}}}^{p} \Bigg\},$$

and the elements $(A_i)_{\alpha\beta}$, $i = 2, 3, 4$, are given by

$$(A_2)_{\alpha\beta} = \begin{cases} \mathcal{F}_{\alpha\beta} & \alpha = 1,\ldots,p,\ \beta = 1,\ldots,p,\ \beta \neq \{\varrho^1_{m_1}\},\ldots,\{\varrho^n_{m_n}\} \\ \mathcal{G}^k_{\alpha\beta} & \alpha = 1,\ldots,p,\ \beta = \{\varrho^k_{m_k}\}(k = 1,2,\ldots,p) \end{cases},$$

$$(4.25)$$

$$(A_3)_{\alpha\beta} = \begin{cases} \mathcal{F}_{\alpha\beta} & \alpha = 1,\ldots,p,\ \alpha \neq \{\rho^1_{m_1}\},\ldots,\{\rho^{n-1}_{m_{n-1}}\},\ \beta = 1,\ldots,p \\ \dfrac{1}{\beta+1}\mathcal{H}^k_{\alpha\beta} & \alpha = \{\rho^k_{m_k}\}(k = 1,2,\ldots,p-1),\ \beta = 1,\ldots,p \end{cases},$$

$$(4.26)$$

and

$$(A_4)_{\alpha\beta} = \begin{cases} \mathcal{F}_{\alpha\beta} & \begin{array}{l} \alpha = 1,\ldots,p,\ \alpha \neq \{\rho^1_{m_1}\},\ldots,\{\rho^{n-1}_{m_{n-1}}\}, \\ \beta = 1,\ldots,p,\ \beta \neq \{\varrho^1_{m_1}\},\ldots,\{\varrho^n_{m_n}\} \end{array} \\ \mathcal{G}^k_{\alpha\beta} & \begin{array}{l} \alpha = 1,\ldots,p,\ \alpha \neq \{\rho^1_{m_1}\},\ldots,\{\rho^{n-1}_{m_{n-1}}\}, \\ \beta = \{\varrho^k_{m_k}\}\,(k = 1,2,\ldots,n) \end{array} \\ \dfrac{1}{\beta+1}\mathcal{H}^k_{\alpha\beta} & \alpha = \{\rho^k_{m_k}\}\,(k = 1,\ldots n-1),\ \beta = 1,\ldots,p \end{cases},\quad (4.27)$$

respectively.

Thus by using (4.24), the function $G^{(p)}$ (4.18) becomes

$$G^{(p)}(\{\lambda_k\}_{(p,n)}, \mu_1,\ldots,\mu_p, i_{p+1},\ldots,i_n)$$

$$= (-1)^p 2^{\frac{(n-p)(n-p-1)}{2}} \sum_{j=1}^{4} \mathcal{T}_j$$

$$\times \prod_{l=1}^{p} \prod_{k=p+1}^{n} a(\mu_l, \xi_{i_k}) \det \mathcal{B}_{n-p}(\lambda_{p+1},\ldots,\lambda_n; \xi_{i_{p+1}},\ldots,\xi_{i_n})$$

$$\equiv \sum_{j=1}^{4} G^{(p)}_j(\{\lambda_k\}_{(p,n)}, \mu_1,\ldots,\mu_p, i_{p+1},\ldots,i_n). \tag{4.28}$$

19

## 4.2  $m \geq p+1$

Then we compute the intermediate functions $G^{(m)}$ for $m \geq p+1$. Similar to the $m \leq p$ case, inserting a complete set and noticing (4.28), we have

$$
\begin{aligned}
& G^{(m)}(\{\lambda_k\}_{(p,n)}, \mu_1, \ldots, \mu_m, i_{m+1}, \ldots, i_n) \\
& = \sum_{j \neq i_{m+1}, \ldots, i_n}^{N} \langle 0| \overset{\overleftarrow{n}}{\prod_{k=m+1}} E^{31}_{(i_k)} \tilde{B}_1(\mu_m) \overset{\overrightarrow{m+q}}{\prod_{k=m+1}} E^{13}_{(i_k)} E^{13}_{(j)} \overset{\overrightarrow{n}}{\prod_{m+q+1}} E^{13}_{(i_k)} |0\rangle \\
& \quad \times G^{(m-1)}(\{\lambda_k\}_{(p,n)}, \mu_1, \ldots, \mu_{m-1}, i_{m+1}, \ldots, i_{m+q}, j, i_{m+q+1} \ldots, i_n) \\
& = \sum_{j=1}^{4} G^{(m)}_j(\{\lambda_k\}_{(p,n)}, \mu_1, \ldots, \mu_m, i_{m+1}, \ldots, i_n),
\end{aligned}
\tag{4.29}
$$

where $G^{(m)}_j$'s correspond to $G^{(p)}_j$'s in (4.28), respectively.

We first compute $G^{(m)}_1$. With the help of the expression of $\tilde{B}_1$ (3.12), we have

$$
\begin{aligned}
& \langle 0| \overset{\overleftarrow{n}}{\prod_{k=m+1}} E^{31}_{(i_k)} \tilde{B}_1(\mu_m) \overset{\overrightarrow{m+q}}{\prod_{k=m+1}} E^{13}_{(i_k)} E^{13}_{(j)} \overset{\overrightarrow{n}}{\prod_{m+q+1}} E^{13}_{(i_k)} |0\rangle \\
& = -(-1)^q 2^{-(n-m)} \cdot b(\mu_m, \xi_j) \prod_{l=m+1}^{n} a(\mu_m, \xi_{i_l}) \prod_{k \neq j}^{N} a^{-1}(\xi_k, \xi_j).
\end{aligned}
\tag{4.30}
$$

When $m = p+1$, by using the expressions (4.28) and (4.30), the intermediate function $G^{(p+1)}_1$ is given by

$$
\begin{aligned}
& G^{(p+1)}_1(\{\lambda_k\}_{(p,n)}, \mu_1, \ldots, \mu_{p+1}, i_{p+2}, \ldots, i_n) \\
& = \sum_{j \neq i_{p+2}, \ldots, i_n}^{N} \langle 0| \overset{\overleftarrow{n}}{\prod_{k=p+2}} E^{31}_{(i_k)} \tilde{B}_1(\mu_{p+1}) \overset{\overrightarrow{p+q+1}}{\prod_{k=p+2}} E^{13}_{(i_k)} E^{13}_{(j)} \overset{\overrightarrow{n}}{\prod_{p+q+2}} E^{13}_{(i_k)} |0\rangle \\
& \quad \times G^{(p)}_1(\{\lambda_k\}_{(p,n)}, \mu_1, \ldots, \mu_p, i_{p+2}, \ldots, i_{p+q+1}, j, i_{p+q+2} \ldots, i_n) \\
& = (-1)^m 2^{\frac{(n-p)(n-p-1) - 2(n-p-1)}{2}} \det \mathcal{F}(\lambda_1, \ldots, \lambda_p; \mu_1, \ldots, \mu_p) \\
& \quad \times \det \mathcal{B}^{(p+1)}_{n-p}(\lambda_{p+1}, \ldots, \lambda_n; \mu_{p+1}; \xi_{i_{p+2}}, \ldots, \xi_{i_n}),
\end{aligned}
\tag{4.31}
$$

where the matrix elements $(\mathcal{B}^{(m)}_{n-p})_{\alpha\beta}$ $(p+1 \leq \alpha, \beta \leq n)$

$$
(\mathcal{B}^{(p+1)}_{n-p})_{\alpha\beta} = \prod_{\epsilon=1}^{\beta-1} a(\mu_\epsilon, \xi_{i_\beta})(\mathcal{B}_{n-p})_{\alpha\beta}, \qquad \text{for } p+1 < \beta \leq n,
$$

$$(\mathcal{B}_{n-p}^{(p+1)})_{\alpha\,p+1} \;=\; \sum_{j\neq i_{p+2},\ldots,i_n} b(\mu_{p+1},\xi_j)b(\lambda_\alpha,\xi_j) \prod_{\gamma=p+1}^{\alpha-1} a(\lambda_\gamma,\xi_j) \prod_{\epsilon=1}^{p} a(\mu_\epsilon,\xi_j) \prod_{k\neq j}^{N} a^{-1}(\xi_k,\xi_j).$$

$$(4.32)$$

As a element of the matrix $\mathcal{B}_{n-p}^{(p+1)}$, one finds if we take $j = i_{p+2},\ldots,i_n$ in the sum of (4.32), the added terms will not contribute to the determinant. Therefore we may rewrite $(\mathcal{B}_{n-p}^{(p+1)})_{\alpha\,p+1}$ as

$$(\mathcal{B}_{n-p}^{(p+1)})_{\alpha\,p+1} \;=\; \sum_{j=1}^{N} \frac{\eta}{\mu_{p+1}-\xi_j+\eta} \frac{\eta}{\lambda_\alpha-\xi_j+\eta} \prod_{\gamma=p+1}^{\alpha-1} \frac{\lambda_\gamma-\xi_j}{\lambda_\gamma-\xi_j+\eta}$$
$$\times \prod_{\epsilon=1}^{p} \frac{\mu_\epsilon-\xi_j}{\mu_\epsilon-\xi_j+\eta} \prod_{k\neq j}^{N} \frac{\xi_k-\xi_j+\eta}{\xi_k-\xi_j}.$$

$$(4.33)$$

Then by using the properties of rational function again, we construct the function $(\mathcal{N}_1)_{\alpha\beta}$ $(p+1 \leq \alpha,\beta \leq n)$

$$(\mathcal{N}_1)_{\alpha\beta} \;=\; \frac{\eta}{\lambda_\alpha-\mu_\beta} \prod_{\gamma=p+1}^{\alpha-1} \frac{\lambda_\gamma-\mu_\beta-\eta}{\lambda_\gamma-\mu_\beta} \prod_{\epsilon=1}^{\beta-1} \frac{\mu_\epsilon-\mu_\beta-\eta}{\mu_\epsilon-\mu_\beta}$$
$$\times \left[ \prod_{j=p+1}^{n} \frac{\mu_\beta-\lambda_j^{(1)}}{\mu_\beta-\lambda_j^{(1)}+\eta} - \prod_{k=1}^{N} \frac{\mu_\beta-\xi_k}{\mu_\beta-\xi_k+\eta} \right]$$
$$+ \sum_{j=p+1}^{n} \left[ \frac{\eta}{\mu_\beta-\lambda_j^{(1)}+\eta} \frac{\eta}{\lambda_\alpha-\lambda_j^{(1)}+\eta} \prod_{\gamma=p+1}^{\alpha-1} \frac{\lambda_\gamma-\lambda_j^{(1)}}{\lambda_\gamma-\lambda_j^{(1)}+\eta} \right.$$
$$\left. \times \prod_{\epsilon=1}^{\beta-1} \frac{\mu_\epsilon-\lambda_j^{(1)}}{\mu_\epsilon-\lambda_j^{(1)}+\eta} \prod_{k\neq j} \frac{\lambda_k^{(1)}-\lambda_j^{(1)}+\eta}{\lambda_k^{(1)}-\lambda_j^{(1)}} \right]$$
$$+ \sum_{\epsilon=1}^{\beta-1} \frac{\eta}{\lambda_\alpha-\mu_\epsilon} \frac{\eta}{\mu_\epsilon-\mu_\beta} \prod_{\gamma=p+1}^{\alpha-1} \frac{\lambda_\gamma-\mu_\epsilon-\eta}{\lambda_\gamma-\mu_\epsilon} \prod_{\varepsilon=1,\neq\epsilon}^{\beta-1} \frac{\mu_\varepsilon-\mu_\epsilon-\eta}{\mu_\varepsilon-\mu_\epsilon}$$
$$\times \left[ \prod_{j=p+1}^{n} \frac{\mu_\epsilon-\lambda_j^{(1)}}{\mu_\epsilon-\lambda_j^{(1)}+\eta} - \prod_{k=1}^{N} \frac{\mu_\epsilon-\xi_k}{\mu_\epsilon-\xi_k+\eta} \right]. \qquad (4.34)$$

Here as before, one may prove $(\mathcal{B}_{n-p}^{(p+1)})_{\alpha\,p+1} = (\mathcal{N}_1)_{\alpha\,p+1}$. Moreover, with a similar procedure, one may prove that for any $p+1 \leq m \leq n$, the function $G_1^{(m)}$ can be written as

$$G_1^{(m)}(\{\lambda_k\}_{(p,n)}, \mu_1,\ldots,\mu_m, i_{m+1},\ldots,i_n)$$
$$= (-1)^m 2^{\frac{(n-p)(n-p-1)-(m-p)(2n-m-p-1)}{2}} \det\mathcal{F}(\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_p)$$
$$\times \det\mathcal{B}_{n-p}(\lambda_{p+1},\ldots,\lambda_n;\xi_{i_{p+1}},\ldots,\xi_{i_n}), \qquad (4.35)$$

21

where the matrix elements $(\mathcal{B}_{n-p}^{(m)})_{\alpha\beta}$ $(p+1 \leq \alpha, \beta \leq n)$

$$
(\mathcal{B}_{n-p}^{(m)})_{\alpha\beta} = \prod_{\epsilon=1}^{\beta-1} a(\mu_\epsilon, \xi_{i_\beta})(\mathcal{B}_p)_{\alpha\beta}, \qquad \text{for } m < \beta \leq n,
$$

$$
(\mathcal{B}_{n-p}^{(m)})_{\alpha\beta} = (\mathcal{N}_1)_{\alpha\beta}, \qquad \text{for } p+1 \leq \beta \leq m. \qquad (4.36)
$$

Therefore when $m = n$, we obtain

$$
G_1^{(n)}(\{\lambda_j\}_{(p,n)}, \{\mu_k\}_{(p,n)}) = (-1)^n \det\mathcal{F}(\lambda_1, \ldots, \lambda_p; \mu_1, \ldots, \mu_p)
$$
$$
\times \det\mathcal{N}_1(\lambda_{p+1}, \ldots, \lambda_n; \mu_{p+1}, \ldots, \mu_n). \qquad (4.37)
$$

Similarly, the function $G_2^{(n)}$ is given by

$$
G_2^{(n)}(\{\lambda_j\}_{(p,n)}, \{\mu_k\}_{(p,n)}) = (-1)^n \sum_{k,l_k,\varrho_{m_k}^k}' \sum_{\substack{j_1,\ldots,j_p=1 \\ j_1\neq\ldots\neq j_p}}^{p} (-1)^{\tau(j_1 j_2 \ldots j_p)} (A_2)_{1\,j_1}(A_2)_{2\,j_2} \ldots (A_2)_{p\,j_p}
$$
$$
\times \det\mathcal{N}_2(\lambda_{p+1}, \ldots, \lambda_n; \mu_{p+1}, \ldots, \mu_n) \qquad (4.38)
$$

with

$$
(\mathcal{N}_2)_{\alpha\beta} = \frac{\eta}{\lambda_\alpha - \mu_\beta} \prod_{e=1}^{k} \left( \frac{\mu_e - \mu_\beta}{\mu_e - \mu_\beta - \eta} \right)^{l_e-1} \prod_{\epsilon=k+1}^{\beta-1} \frac{\mu_\epsilon - \mu_\beta - \eta}{\mu_\epsilon - \mu_\beta} \prod_{\gamma=p+1}^{\alpha-1} \frac{\lambda_\gamma - \mu_\beta - \eta}{\lambda_\gamma - \mu_\beta}
$$
$$
\times \prod_{f'=1}^{p} \prod_{t=1}^{k} \prod_{t'=1}^{l_t} \left[ 1 + \delta_{j_{f'}\,\varrho_{t'}^t} \left( \frac{\lambda_{f'} - \mu_\beta - \eta}{\lambda_{f'} - \mu_\beta} - 1 \right) \right]
$$
$$
\times \left[ \prod_{j=p+1}^{n} \frac{\mu_\beta - \lambda_j^{(1)}}{\mu_\beta - \lambda_j^{(1)} + \eta} - \prod_{l=1}^{N} \frac{\mu_\beta - \xi_l}{\mu_\beta - \xi_l + \eta} \right]
$$
$$
+ \sum_{\theta=p+1}^{n} \frac{\eta}{\mu_\beta - \lambda_\theta^{(1)} + \eta} \frac{\eta}{\lambda_\alpha - \lambda_\theta^{(1)} + \eta} \prod_{e=1}^{k} \left( \frac{\mu_e - \lambda_\theta^{(1)} + \eta}{\mu_e - \lambda_\theta^{(1)}} \right)^{l_e-1}
$$
$$
\times \prod_{\epsilon=k+1}^{\beta-1} \frac{\mu_\epsilon - \lambda_\theta^{(1)}}{\mu_\epsilon - \lambda_\theta^{(1)} + \eta} \prod_{\gamma=p+1}^{\alpha-1} \frac{\lambda_\gamma - \lambda_\theta^{(1)}}{\lambda_\gamma - \lambda_\theta^{(1)} + \eta} \prod_{\vartheta=p+1,\neq\theta}^{n} \frac{\lambda_\vartheta - \lambda_\theta^{(1)} + \eta}{\lambda_\vartheta - \lambda_\theta^{(1)}}
$$
$$
\times \prod_{f'=1}^{p} \prod_{t=1}^{k} \prod_{t'=1}^{l_t} \left[ 1 + \delta_{j_{f'}\,\varrho_{t'}^t} \left( \frac{\lambda_{f'} - \lambda_\theta^{(1)}}{\lambda_{f'} - \lambda_\theta^{(1)} + \eta} - 1 \right) \right]
$$
$$
+ \sum_{\epsilon=k+1}^{\beta-1} \frac{\eta}{\mu_\epsilon - \mu_\beta} \frac{\eta}{\lambda_\alpha - \mu_\epsilon} \prod_{e=1}^{k} \left( \frac{\mu_e - \mu_\epsilon}{\mu_e - \mu_\epsilon - \eta} \right)^{l_e-1} \prod_{\varepsilon=k+1,\neq\epsilon}^{\beta-1} \frac{\mu_\varepsilon - \mu_\epsilon - \eta}{\mu_\varepsilon - \mu_\epsilon}
$$
$$
\times \prod_{\gamma=p+1}^{\alpha-1} \frac{\lambda_\gamma - \mu_\epsilon - \eta}{\lambda_\gamma - \mu_\epsilon} \left[ \prod_{j=p+1}^{n} \frac{\mu_\epsilon - \lambda_j^{(1)}}{\mu_\epsilon - \lambda_j^{(1)} + \eta} - \prod_{l=1}^{N} \frac{\mu_\epsilon - \xi_l}{\mu_\epsilon - \xi_l + \eta} \right]
$$

22

$$\times \prod_{f'=1}^{p} \prod_{t=1}^{k} \prod_{t'=1}^{l_t} \left[ 1 + \delta_{j_{f'} \varrho_{t'}^{t}} \left( \frac{\lambda_{f'} - \mu_\epsilon - \eta}{\lambda_{f'} - \mu_\epsilon} - 1 \right) \right]$$

$$+ \sum_{e=1}^{k} g_2(\mu_\beta, l_e), \tag{4.39}$$

where the function $g_2(\mu_\beta, l_e) = 0$ when $l_e = 1$; when $l_e = 0$,

$$\begin{aligned}
g_2(\mu_\beta, l_e) &= \frac{\eta}{\mu_e - \mu_\beta} \frac{\eta}{\lambda_\alpha - \mu_e} \prod_{e'=1, \neq e}^{k} \left( \frac{\mu_{e'} - \mu_e}{\mu_{e'} - \mu_e - \eta} \right)^{l_{e'}-1} \prod_{\epsilon=k+1}^{\beta-1} \frac{\mu_\epsilon - \mu_e - \eta}{\mu_\epsilon - \mu_e} \\
&\quad \times \prod_{\gamma=p+1}^{\alpha-1} \frac{\lambda_\gamma - \mu_e - \eta}{\lambda_\gamma - \mu_e} \left[ \prod_{j=p+1}^{n} \frac{\mu_e - \lambda_j^{(1)}}{\mu_e - \lambda_j^{(1)} + \eta} - \prod_{l=1}^{N} \frac{\mu_e - \xi_l}{\mu_e - \xi_l + \eta} \right] \\
&\quad \times \prod_{f'=1}^{p} \prod_{t=1}^{k} \prod_{t'=1}^{l_t} \left[ 1 + \delta_{j_{f'} \varrho_{t'}^{t}} \left( \frac{\lambda_{f'} - \mu_e - \eta}{\lambda_{f'} - \mu_e} - 1 \right) \right]
\end{aligned} \tag{4.40}$$

and when $l_e \geq 2$,

$$\begin{aligned}
g_2(\mu_\beta, l_e) &= -\sum_{k=0}^{l_e-2} \frac{1}{k!} \frac{1}{(\mu_\beta - \mu_e + \eta)^{l_e-k-1}} \frac{d^k}{d\chi^k} \left\{ (\chi - \mu_e)^{l_e-1} \frac{\eta}{\lambda_\alpha - \chi} \right. \\
&\quad \times \prod_{e'=1, \neq e}^{k} \left( \frac{\mu_{e'} - \chi}{\mu_{e'} - \chi - \eta} \right)^{l_{e'}-1} \prod_{\epsilon=k+1}^{\beta-1} \frac{\mu_\epsilon - \chi - \eta}{\mu_\epsilon - \chi} \prod_{\gamma=p+1}^{\alpha-1} \frac{\lambda_\gamma - \chi - \eta}{\lambda_\gamma - \chi} \\
&\quad \times \prod_{f'=1}^{p} \prod_{t=1}^{k} \prod_{t'=1}^{l_t} \left[ 1 + \delta_{j_{f'} \varrho_{t'}^{t}} \left( \frac{\lambda_{f'} - \chi - \eta}{\lambda_{f'} - \chi} - 1 \right) \right] \\
&\quad \times \left. \left[ \prod_{j=p+1}^{n} \frac{\chi - \lambda_j^{(1)}}{\chi - \lambda_j^{(1)} + \eta} - \prod_{l=1}^{N} \frac{\chi - \xi_l}{\chi - \xi_l + \eta} \right] \right\}_{\chi=\mu_e-\eta}.
\end{aligned} \tag{4.41}$$

The function $G_3^{(n)}$ is given by

$$\begin{aligned}
G_3^{(n)}(\{\lambda_j\}_{(p,n)}, \{\mu_k\}_{(p,n)}) &= (-1)^n \sum_{k,l_k,\rho_{m_k}^k}{}'' \det A_3(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(p)}; \mu_{\sigma'(1)}, \ldots, \mu_{\sigma'(p)}) \\
&\quad \times \det \mathcal{N}_3(\lambda_{p+1}, \ldots, \lambda_n; \mu_{p+1}, \ldots, \mu_n)
\end{aligned} \tag{4.42}$$

with

$$\begin{aligned}
(\mathcal{N}_3)_{\alpha\beta} &= \frac{\eta}{\lambda_\alpha - \mu_\beta} \prod_{e=1}^{k} \left( \frac{\lambda_e - \mu_\beta}{\lambda_e - \mu_\beta - \eta} \right)^{l_e} \prod_{t=1}^{k} \prod_{t'=1}^{l_t} \frac{\lambda_{\rho_{t'}^{t}} - \mu_\beta - \eta}{\lambda_{\rho_{t'}^{t}} - \mu_\beta} \prod_{\gamma=p+1}^{\alpha-1} \frac{\lambda_\gamma - \mu_\beta - \eta}{\lambda_\gamma - \mu_\beta} \\
&\quad \times \prod_{\epsilon=1}^{\beta-1} \frac{\mu_\epsilon - \mu_\beta - \eta}{\mu_\epsilon - \mu_\beta} \left[ \prod_{j=p+1}^{n} \frac{\mu_\beta - \lambda_j^{(1)}}{\mu_\beta - \lambda_j^{(1)} + \eta} - \prod_{l=1}^{N} \frac{\mu_\beta - \xi_l}{\mu_\beta - \xi_l + \eta} \right]
\end{aligned}$$

23

$$
+ \sum_{\theta=p+1}^{n} \frac{\eta}{\mu_\beta - \lambda_\theta^{(1)} + \eta} \frac{\eta}{\lambda_\alpha - \lambda_\theta^{(1)} + \eta} \prod_{\epsilon=1}^{\beta-1} \frac{\mu_\epsilon - \lambda_\theta^{(1)}}{\mu_\epsilon - \lambda_\theta^{(1)} + \eta} \prod_{\gamma=p+1}^{\alpha-1} \frac{\lambda_\gamma - \lambda_\theta^{(1)}}{\lambda_\gamma - \lambda_\theta^{(1)} + \eta}
$$

$$
\times \prod_{e=1}^{k} \left( \frac{\mu_e - \lambda_\theta^{(1)} + \eta}{\mu_e - \lambda_\theta^{(1)}} \right)^{l_e} \prod_{t=1}^{k} \prod_{t'=1}^{l_t} \frac{\lambda_{\rho_{t'}^t} - \lambda_\theta^{(1)}}{\lambda_{\rho_{t'}^t} - \lambda_\theta^{(1)} + \eta} \prod_{\vartheta=p+1, \neq \theta}^{n} \frac{\lambda_\vartheta - \lambda_\theta^{(1)} + \eta}{\lambda_\vartheta - \lambda_\theta^{(1)}}
$$

$$
+ \sum_{\epsilon=k+1}^{\beta-1} \frac{\eta}{\mu_\epsilon - \mu_\beta} \frac{\eta}{\lambda_\alpha - \mu_\epsilon} \prod_{e=1}^{k} \left( \frac{\lambda_e - \mu_\epsilon}{\lambda_e - \mu_\epsilon - \eta} \right)^{l_e} \prod_{t=1}^{k} \prod_{t'=1}^{l_t} \frac{\lambda_{\rho_{t'}^t} - \mu_\epsilon - \eta}{\lambda_{\rho_{t'}^t} - \mu_\epsilon}
$$

$$
\times \prod_{\varepsilon=1, \neq \epsilon}^{\beta-1} \frac{\mu_\varepsilon - \mu_\epsilon - \eta}{\mu_\varepsilon - \mu_\epsilon} \prod_{\gamma=p+1}^{\alpha-1} \frac{\lambda_\gamma - \mu_\epsilon - \eta}{\lambda_\gamma - \mu_\epsilon} \left[ \prod_{j=p+1}^{n} \frac{\mu_\epsilon - \lambda_j^{(1)}}{\mu_\epsilon - \lambda_j^{(1)} + \eta} - \prod_{l=1}^{N} \frac{\mu_\epsilon - \xi_l}{\mu_\epsilon - \xi_l + \eta} \right]
$$

$$
+ \sum_{e=1}^{k} g_3(\mu_\beta, l_e), \tag{4.43}
$$

where the function $g_3(\mu_\beta, l_e)$ is given as follows. i.) when $\prod_{t=1}^{k} \prod_{t'=1}^{l_t} \delta_{e\, \rho_{t'}^t} = 0$,

$$
g_3(\mu_\beta, l_e) = -\sum_{k=0}^{l_e-1} \frac{1}{k!} \frac{1}{(\mu_\beta - \lambda_e + \eta)^{l_e-k}} \frac{d^k}{d\chi^k} \Bigg\{ (\chi - \lambda_e)^{l_e} \frac{\eta}{\lambda_\alpha - \chi} \prod_{e'=1, \neq e}^{k} \left( \frac{\lambda_e - \chi}{\lambda_e - \chi - \eta} \right)^{l_e}
$$

$$
\times \prod_{t=1}^{k} \prod_{t'=1}^{l_t} \frac{\lambda_{\rho_{t'}^t} - \chi - \eta}{\lambda_{\rho_{t'}^t} - \chi} \prod_{\gamma=p+1}^{\alpha-1} \frac{\lambda_\gamma - \chi - \eta}{\lambda_\gamma - \chi}
$$

$$
\times \prod_{\epsilon=1}^{\beta-1} \frac{\mu_\epsilon - \chi - \eta}{\mu_\epsilon - \chi} \left[ \prod_{j=p+1}^{n} \frac{\chi - \lambda_j^{(1)}}{\chi - \lambda_j^{(1)} + \eta} - \prod_{l=1}^{N} \frac{\chi - \xi_l}{\chi - \xi_l + \eta} \right] \Bigg\}_{\chi = \lambda_e - \eta}, \tag{4.44}
$$

ii.) when $\prod_{t=1}^{k} \prod_{t'=1}^{l_t} \delta_{e\, \rho_{t'}^t} = 1$ and $l_e = 1$, $g_3(\mu_\beta, l_e) = 0$ and iii.) when there is an index $\hat{t}$ ($\hat{t} \in \{1, \ldots k\}$) and $\hat{t'}$ ($\hat{t'} \in \{1, \ldots l_{\hat{t}}\}$) such that $\rho_{\hat{t'}}^{\hat{t}} = e$, and $l_e \geq 2$,

$$
g_3(\mu_\beta, l_e) = -\sum_{k=0}^{l_e-2} \frac{1}{k!} \frac{1}{(\mu_\beta - \lambda_e + \eta)^{l_e-k-1}} \frac{d^k}{d\chi^k} \Bigg\{ (\chi - \lambda_e)^{l_e-1} \frac{\eta}{\lambda_\alpha - \chi} \prod_{e'=1, \neq e}^{k} \left( \frac{\lambda_e - \chi}{\lambda_e - \chi - \eta} \right)^{l_e}
$$

$$
\times \prod_{t=1, \neq \hat{t}}^{k} \prod_{t'=1, \neq \hat{t'}}^{l_t} \frac{\lambda_{\rho_{t'}^t} - \chi - \eta}{\lambda_{\rho_{t'}^t} - \chi} \prod_{\gamma=p+1}^{\alpha-1} \frac{\lambda_\gamma - \chi - \eta}{\lambda_\gamma - \chi}
$$

$$
\times \prod_{\epsilon=1}^{\beta-1} \frac{\mu_\epsilon - \chi - \eta}{\mu_\epsilon - \chi} \left[ \prod_{j=p+1}^{n} \frac{\chi - \lambda_j^{(1)}}{\chi - \lambda_j^{(1)} + \eta} - \prod_{l=1}^{N} \frac{\chi - \xi_l}{\chi - \xi_l + \eta} \right] \Bigg\}_{\chi = \lambda_e - \eta}. \tag{4.45}
$$

The function $G_4^{(n)}$ is given by

$$
G_4^{(n)}(\{\lambda_j\}_{(p,n)}, \{\mu_k\}_{(p,n)})
$$

$$
= (-1)^n \sideset{}{'}\sum_{k,l_k,\varrho_{m_k}^k} \sideset{}{''}\sum_{k',l_{k'}',\rho_{m_{k'}'}^{k'}} \sum_{\substack{j_1,\ldots,j_p=1 \\ j_1 \neq \ldots \neq j_p}}^{p} (-1)^{\tau(j_1 j_2 \ldots j_p)} (A_4)_{1\, j_1} (A_4)_{2\, j_2} \ldots (A_4)_{p\, j_p}
$$

24

$$\times \det \mathcal{N}_4(\lambda_{p+1}, \ldots, \lambda_n; \mu_{p+1}, \ldots, \mu_n) \tag{4.46}$$

with

$$
\begin{aligned}
(\mathcal{N}_4)_{\alpha\beta} \;=\; & \frac{\eta}{\lambda_\alpha - \mu_\beta} \prod_{e=1}^{k} \left( \frac{\lambda_e - \mu_\beta}{\lambda_e - \mu_\beta - \eta} \right)^{l'_e} \prod_{t=1}^{k'} \prod_{t'=1}^{l'_t} \frac{\lambda_{\rho^t_{t'}} - \mu_\beta - \eta}{\lambda_{\rho^t_{t'}} - \mu_\beta} \prod_{\gamma=p+1}^{\alpha-1} \frac{\lambda_\gamma - \mu_\beta - \eta}{\lambda_\gamma - \mu_\beta} \\
& \times \prod_{\epsilon=1}^{\beta-1} \frac{\mu_\epsilon - \mu_\beta - \eta}{\mu_\epsilon - \mu_\beta} \left[ \prod_{j=p+1}^{n} \frac{\mu_\beta - \lambda_j^{(1)}}{\mu_\beta - \lambda_j^{(1)} + \eta} - \prod_{l=1}^{N} \frac{\mu_\beta - \xi_l}{\mu_\beta - \xi_l + \eta} \right] \\
& \times \prod_{t=1}^{k} \prod_{\substack{s=1 \\ s \neq \{\rho^1_{m_1}\}, \ldots, \{\rho^k_{m_k}\}}}^{p} \prod_{f'=1}^{p} \prod_{t'=1}^{l_t} \left[ 1 + \delta_{f' s} \, \delta_{j_{f'} \, \varrho^t_{t'}} \left( \frac{\mu_t - \mu_\beta}{\mu_t - \mu_\beta - \eta} - 1 \right) \right] \\
& \times \prod_{t=1}^{k} \prod_{\substack{s=1 \\ s \neq \{\rho^1_{m_1}\}, \ldots, \{\rho^k_{m_k}\}}}^{p} \prod_{f'=1}^{p} \prod_{t'=1}^{l_t} \left[ 1 + \delta_{f' s} \, \delta_{j_{f'} \, \varrho^t_{t'}} \left( \frac{\lambda_s - \mu_\beta - \eta}{\lambda_s - \mu_\beta} - 1 \right) \right] \\
+ & \sum_{\theta=p+1}^{n} \frac{\eta}{\mu_\beta - \lambda_\theta^{(1)} + \eta} \frac{\eta}{\lambda_\alpha - \lambda_\theta^{(1)} + \eta} \prod_{\epsilon=1}^{\beta-1} \frac{\mu_\epsilon - \lambda_\theta^{(1)}}{\mu_\epsilon - \lambda_\theta^{(1)} + \eta} \prod_{\gamma=p+1}^{\alpha-1} \frac{\lambda_\gamma - \lambda_\theta^{(1)}}{\lambda_\gamma - \lambda_\theta^{(1)} + \eta} \\
& \times \prod_{e=1}^{k} \left( \frac{\mu_e - \lambda_\theta^{(1)} + \eta}{\mu_e - \lambda_\theta^{(1)}} \right)^{l'_e} \prod_{t=1}^{k'} \prod_{t'=1}^{l'_t} \frac{\lambda_{\rho^t_{t'}} - \lambda_\theta^{(1)}}{\lambda_{\rho^t_{t'}} - \lambda_\theta^{(1)} + \eta} \prod_{\vartheta=p+1, \neq \theta}^{n} \frac{\lambda_\vartheta - \lambda_\theta^{(1)} + \eta}{\lambda_\vartheta - \lambda_\theta^{(1)}} \\
& \times \prod_{t=1}^{k} \prod_{\substack{s=1 \\ s \neq \{\rho^1_{m_1}\}, \ldots, \{\rho^k_{m_k}\}}}^{p} \prod_{f'=1}^{p} \prod_{t'=1}^{l_t} \left[ 1 + \delta_{f' s} \, \delta_{j_{f'} \, \varrho^t_{t'}} \left( \frac{\mu_t - \lambda_\theta^{(1)} + \eta}{\mu_t - \lambda_\theta^{(1)}} - 1 \right) \right] \\
& \times \prod_{t=1}^{k} \prod_{\substack{s=1 \\ s \neq \{\rho^1_{m_1}\}, \ldots, \{\rho^k_{m_k}\}}}^{p} \prod_{f'=1}^{p} \prod_{t'=1}^{l_t} \left[ 1 + \delta_{f' s} \, \delta_{j_{f'} \, \varrho^t_{t'}} \left( \frac{\lambda_s - \lambda_\theta^{(1)}}{\lambda_s - \lambda_\theta^{(1)} + \eta} - 1 \right) \right] \\
+ & \sum_{\epsilon=k+1}^{\beta-1} \frac{\eta}{\mu_\epsilon - \mu_\beta} \frac{\eta}{\lambda_\alpha - \mu_\epsilon} \prod_{e=1}^{k} \left( \frac{\lambda_e - \mu_\epsilon}{\lambda_e - \mu_\epsilon - \eta} \right)^{l'_e} \prod_{t=1}^{k'} \prod_{t'=1}^{l'_t} \frac{\lambda_{\rho^t_{t'}} - \mu_\epsilon - \eta}{\lambda_{\rho^t_{t'}} - \mu_\epsilon} \\
& \times \prod_{\varepsilon=1, \neq \epsilon}^{\beta-1} \frac{\mu_\varepsilon - \mu_\epsilon - \eta}{\mu_\varepsilon - \mu_\epsilon} \prod_{\gamma=p+1}^{\alpha-1} \frac{\lambda_\gamma - \mu_\epsilon - \eta}{\lambda_\gamma - \mu_\epsilon} \\
& \times \left[ \prod_{j=p+1}^{n} \frac{\mu_\epsilon - \lambda_j^{(1)}}{\mu_\epsilon - \lambda_j^{(1)} + \eta} - \prod_{l=1}^{N} \frac{\mu_\epsilon - \xi_l}{\mu_\epsilon - \xi_l + \eta} \right] \\
& \times \prod_{t=1}^{k} \prod_{\substack{s=1 \\ s \neq \{\rho^1_{m_1}\}, \ldots, \{\rho^k_{m_k}\}}}^{p} \prod_{f'=1}^{p} \prod_{t'=1}^{l_t} \left[ 1 + \delta_{f' s} \, \delta_{j_{f'} \, \varrho^t_{t'}} \left( \frac{\mu_t - \mu_\epsilon}{\mu_t - \mu_\epsilon - \eta} - 1 \right) \right]
\end{aligned}
$$

$$\times \prod_{\substack{t=1}}^{k} \prod_{\substack{s=1 \\ s\neq\{\rho_{m_1}^1\},...,\{\rho_{m_k}^k\}}}^{p} \prod_{f'=1}^{p} \prod_{t'=1}^{l_t} \left[1 + \delta_{f's}\,\delta_{j_{f'}\,\varrho_{t'}^t}\left(\frac{\lambda_s - \mu_\epsilon - \eta}{\lambda_s - \mu_\epsilon} - 1\right)\right]$$

$$+ \sum_{e=1}^{k'} g_4(\mu_\beta, l'_e) + \sum_{t=1}^{k} g'_4(\mu_\beta, l_t), \tag{4.47}$$

where $g_4(\mu_\beta, l'_e)$ is given as follows. i.) when $\prod_{t=1}^{k'}\prod_{t'=1}^{l'_t} \delta_{e\,\rho_{t'}^t} = 0$,

$$
\begin{aligned}
g_4(\mu_\beta, l'_e) &= -\sum_{k=0}^{l'_e-1} \frac{1}{k!} \frac{1}{(\mu_\beta - \lambda_e + \eta)^{l'_e-k}} \frac{d^k}{d\chi^k} \left\{ (\chi - \lambda_e)^{l'_e} \frac{\eta}{\lambda_\alpha - \chi} \prod_{e'=1,\neq e}^{k'} \left(\frac{\lambda_e - \chi}{\lambda_e - \chi - \eta}\right)^{l'_e} \right. \\
&\times \prod_{t=1}^{k'} \prod_{t'=1}^{l'_t} \frac{\lambda_{\rho_{t'}^t} - \chi - \eta}{\lambda_{\rho_{t'}^t} - \chi} \prod_{\gamma=p+1}^{\alpha-1} \frac{\lambda_\gamma - \chi - \eta}{\lambda_\gamma - \chi} \\
&\times \prod_{\epsilon=1}^{\beta-1} \frac{\mu_\epsilon - \chi - \eta}{\mu_\epsilon - \chi} \left[ \prod_{j=p+1}^{n} \frac{\chi - \lambda_j^{(1)}}{\chi - \lambda_j^{(1)} + \eta} - \prod_{l=1}^{N} \frac{\chi - \xi_l}{\chi - \xi_l + \eta} \right] \\
&\times \prod_{t=1}^{k} \prod_{\substack{s=1 \\ s\neq\{\rho_{m_1}^1\},...,\{\rho_{m_k}^k\}}}^{p} \prod_{f'=1}^{p} \prod_{t'=1}^{l_t} \left[1 + \delta_{f's}\,\delta_{j_{f'}\,\varrho_{t'}^t}\left(\frac{\mu_t - \chi}{\mu_t - \chi - \eta} - 1\right)\right] \\
&\left. \times \prod_{t=1}^{k} \prod_{\substack{s=1 \\ s\neq\{\rho_{m_1}^1\},...,\{\rho_{m_k}^k\}}}^{p} \prod_{f'=1}^{p} \prod_{t'=1}^{l_t} \left[1 + \delta_{f's}\,\delta_{j_{f'}\,\varrho_{t'}^t}\left(\frac{\lambda_s - \chi - \eta}{\lambda_s - \chi} - 1\right)\right] \right\}_{\chi = \lambda_e - \eta},
\end{aligned}
\tag{4.48}
$$

ii.) when $\prod_{t=1}^{k}\prod_{t'=1}^{l'_t} \delta_{e\,\rho_{t'}^t} = 1$ and $l'_e = 1$, $g_4(\mu_\beta, l_e) = 0$ and iii.) when there are indices $\hat{t}$ ($\hat{t} \in \{1, \ldots k\}$) and $\hat{t}'$ ($\hat{t}' \in \{1, \ldots l_{\hat{t}}\}$) such that $\rho_{\hat{t}'}^{\hat{t}} = e$, and $l_e \geq 2$,

$$
\begin{aligned}
g_4(\mu_\beta, l_e) &= -\sum_{k=0}^{l_e-2} \frac{1}{k!} \frac{1}{(\mu_\beta - \lambda_e + \eta)^{l_e-k-1}} \frac{d^k}{d\chi^k} \left\{ (\chi - \lambda_e)^{l_e-1} \frac{\eta}{\lambda_\alpha - \chi} \right. \\
&\times \prod_{e'=1,\neq e}^{k'} \left(\frac{\lambda_e - \chi}{\lambda_e - \chi - \eta}\right)^{l'_e} \prod_{t=1,\neq\hat{t}}^{k'} \prod_{t'=1,\neq\hat{t}'}^{l'_t} \frac{\lambda_{\rho_{t'}^t} - \chi - \eta}{\lambda_{\rho_{t'}^t} - \chi} \prod_{\gamma=p+1}^{\alpha-1} \frac{\lambda_\gamma - \chi - \eta}{\lambda_\gamma - \chi} \\
&\times \prod_{\epsilon=1}^{\beta-1} \frac{\mu_\epsilon - \chi - \eta}{\mu_\epsilon - \chi} \left[ \prod_{j=p+1}^{n} \frac{\chi - \lambda_j^{(1)}}{\chi - \lambda_j^{(1)} + \eta} - \prod_{l=1}^{N} \frac{\chi - \xi_l}{\chi - \xi_l + \eta} \right] \\
&\times \prod_{t=1}^{k} \prod_{\substack{s=1 \\ s\neq\{\rho_{m_1}^1\},...,\{\rho_{m_k}^k\}}}^{p} \prod_{f'=1}^{p} \prod_{t'=1}^{l_t} \left[1 + \delta_{f's}\,\delta_{j_{f'}\,\varrho_{t'}^t}\left(\frac{\mu_t - \chi}{\mu_t - \chi - \eta} - 1\right)\right]
\end{aligned}
$$

$$\times \prod_{\substack{t=1}}^{k} \prod_{\substack{s=1 \\ s\neq\{\rho_{m_1}^1\},...,\{\rho_{m_k}^k\}}}^{p} \prod_{f'=1}^{p} \prod_{t'=1}^{l_t} \left[1 + \delta_{f's}\,\delta_{j_{f'}\,\varrho_{t'}^t}\left(\frac{\lambda_s - \chi - \eta}{\lambda_s - \chi} - 1\right)\right]\Bigg\}\Bigg|_{\chi=\lambda_e-\eta} \quad ;$$

$$(4.49)$$

for the function $g_4'(\mu_\beta, l_t)$, one has : i.) $g_4'(\mu_\beta, l_t) = 0$ when

$$n_t \equiv \sum_{\substack{s=1 \\ s\neq\{\rho_{m_1}^1\},...,\{\rho_{m_k}^k\}}}^{p} \sum_{f'=1}^{p} \sum_{t'=1}^{l_t} \delta_{f's}\,\delta_{j_{f'}\,\varrho_{t'}^t} = 1,$$

ii.) when $n_t = 0$,

$$g_4'(\mu_\beta, l_t) = \frac{\eta}{\mu_t - \mu_\beta}\frac{\eta}{\lambda_\alpha - \mu_t}\prod_{e=1}^{k}\left(\frac{\lambda_e - \mu_t}{\lambda_e - \mu_t - \eta}\right)^{l_e'}\prod_{t=1}^{k'}\prod_{t'=1}^{l_t'}\frac{\lambda_{\rho_{t'}^t} - \mu_t - \eta}{\lambda_{\rho_{t'}^t} - \mu_t}$$

$$\times \prod_{\epsilon=1,\neq t}^{\beta-1}\frac{\mu_\epsilon - \mu_t - \eta}{\mu_\epsilon - \mu_t}\prod_{\gamma=p+1}^{\alpha-1}\frac{\lambda_\gamma - \mu_t - \eta}{\lambda_\gamma - \mu_t}$$

$$\times \left[\prod_{j=p+1}^{n}\frac{\mu_t - \lambda_j^{(1)}}{\mu_t - \lambda_j^{(1)} + \eta} - \prod_{l=1}^{N}\frac{\mu_t - \xi_l}{\mu_t - \xi_l + \eta}\right]$$

$$\times \prod_{\substack{\tau=1,\neq t}}^{k}\prod_{\substack{s=1 \\ s\neq\{\rho_{m_1}^1\},...,\{\rho_{m_k}^k\}}}^{p}\prod_{f'=1}^{p}\prod_{\tau'=1}^{l_\tau}\left[1 + \delta_{f's}\,\delta_{j_{f'}\,\varrho_{\tau'}^\tau}\left(\frac{\mu_\tau - \mu_t}{\mu_\tau - \mu_t - \eta} - 1\right)\right]$$

$$\times \prod_{\substack{\tau=1}}^{k}\prod_{\substack{s=1 \\ s\neq\{\rho_{m_1}^1\},...,\{\rho_{m_k}^k\}}}^{p}\prod_{f'=1}^{p}\prod_{\tau'=1}^{l_\tau}\left[1 + \delta_{f's}\,\delta_{j_{f'}\,\varrho_{\tau'}^\tau}\left(\frac{\lambda_s - \mu_t - \eta}{\lambda_s - \mu_t} - 1\right)\right], (4.50)$$

iii.) when $n_t \geq 2$,

$$g_4'(\mu_\beta, n_t) = -\sum_{k=0}^{n_t-2}\frac{1}{k!}\frac{1}{(\mu_\beta - \mu_e + \eta)^{n_t-k-1}}\frac{d^k}{d\chi^k}\Big\{(\mu_t - \chi)^{-1}$$

$$\times \prod_{\substack{s=1 \\ s\neq\{\rho_{m_1}^1\},...,\{\rho_{m_k}^k\}}}^{p}\prod_{f'=1}^{p}\prod_{t'=1}^{l_t}\left[(\mu_t - \chi - \eta) + \eta\cdot\delta_{f's}\,\delta_{j_{f'}\,\varrho_{t'}^t}\right]$$

$$\times\frac{\eta}{\lambda_\alpha - \chi}\prod_{e=1}^{k}\left(\frac{\lambda_e - \mu_\beta}{\lambda_e - \chi - \eta}\right)^{l_e'}\prod_{t=1}^{k'}\prod_{t'=1}^{l_t'}\frac{\lambda_{\rho_{t'}^t} - \chi - \eta}{\lambda_{\rho_{t'}^t} - \chi}\prod_{\gamma=p+1}^{\alpha-1}\frac{\lambda_\gamma - \chi - \eta}{\lambda_\gamma - \chi}$$

$$\times\prod_{\epsilon=1,\neq t}^{\beta-1}\frac{\mu_\epsilon - \chi - \eta}{\mu_\epsilon - \chi}\left[\prod_{j=p+1}^{n}\frac{\chi - \lambda_j^{(1)}}{\chi - \lambda_j^{(1)} + \eta} - \prod_{l=1}^{N}\frac{\chi - \xi_l}{\chi - \xi_l + \eta}\right]$$

$$\times \prod_{\substack{\tau=1,\neq t}}^{k} \prod_{\substack{s=1 \\ s\neq\{\rho_{m_1}^1\},\ldots,\{\rho_{m_k}^k\}}}^{p} \prod_{f'=1}^{p}\prod_{\tau'=1}^{l_\tau}\left[1+\delta_{f's}\,\delta_{j_{f'}\,\varrho_{\tau'}^\tau}\left(\frac{\mu_\tau-\chi}{\mu_\tau-\chi-\eta}-1\right)\right]$$

$$\times \prod_{\tau=1}^{k} \prod_{\substack{s=1 \\ s\neq\{\rho_{m_1}^1\},\ldots,\{\rho_{m_k}^k\}}}^{p} \prod_{f'=1}^{p}\prod_{\tau'=1}^{l_\tau}\left[1+\delta_{f's}\,\delta_{j_{f'}\,\varrho_{\tau'}^\tau}\left(\frac{\lambda_s-\chi-\eta}{\lambda_s-\chi}-1\right)\right]\Bigg\}\Bigg|_{\chi=\mu_e-\eta} \quad.$$

$$(4.51)$$

From (4.4) and (4.5), we have the following theorem:

**Theorem 2** *Let the spectral parameters $\{\lambda_k\}$ of the Bethe state $|\Omega_N(\{\lambda_k\}_{(p,n)})\rangle$ be solutions of the BAE (2.22). The scalar products $P_n(\{\mu_k\}_{(p,n)},\{\lambda_k\}_{(p,n)})$ defined by (4.1) are represented by*

$$P_n(\{\mu_k\}_{(p,n)},\{\lambda_k\}_{(p,n)})$$
$$= (-1)^n \sum_{\sigma,\sigma'\in\mathcal{S}_n} Y_L(\{\mu_{\sigma'(j)}\},\{\mu_{\sigma'(k)}^{(1)}\})Y_R(\{\lambda_{\sigma(j)}\},\{\lambda_{\sigma(k)}^{(1)}\})$$
$$\times \Big\{\det\mathcal{F}(\lambda_{\sigma(1)},\ldots,\lambda_{\sigma(p)};\mu_{\sigma'(1)},\ldots,\mu_{\sigma'(p)})$$
$$\times \det\mathcal{N}_1(\lambda_{\sigma(p+1)},\ldots,\lambda_{\sigma(n)};\mu_{\sigma'(p+1)},\ldots,\mu_{\sigma'(n)})$$
$$+ \sum_{k,l_k,\varrho_{m_k}^k}{}'\sum_{\substack{j_1,\ldots,j_p=1 \\ j_1\neq\ldots\neq j_p}}^{p}(-1)^{\tau(\sigma'(j_1)\ldots\sigma'(j_p))}(A_2)_{\sigma(1)\,\sigma'(j_1)}\ldots(A_2)_{\sigma(p)\,\sigma'(j_p)}$$
$$\times \det\mathcal{N}_2(\lambda_{\sigma(p+1)},\ldots,\lambda_{\sigma(n)};\mu_{\sigma'(p+1)},\ldots,\mu_{\sigma'(n)})$$
$$+ \sum_{k,l_k,\rho_{m_k}^k}{}''\det A_3(\lambda_{\sigma(1)},\ldots,\lambda_{\sigma(p)};\mu_{\sigma'(1)},\ldots,\mu_{\sigma'(p)})$$
$$\times \det\mathcal{N}_3(\lambda_{\sigma(p+1)},\ldots,\lambda_{\sigma(n)};\mu_{\sigma'(p+1)},\ldots,\mu_{\sigma'(n)})$$
$$+ \sum_{k,l_k,\varrho_{m_k}^k}{}'\sum_{k',l_{k'}',\rho_{m_{k'}'}^{k'}}{}''\sum_{\substack{j_1,\ldots,j_p=1 \\ j_1\neq\ldots\neq j_p}}^{p}(-1)^{\tau(\sigma'(j_1)\ldots\sigma'(j_p))}(A_4)_{\sigma(1)\,\sigma'(j_1)}\ldots(A_2)_{\sigma(p)\,\sigma'(j_p)}$$
$$\times \det\mathcal{N}_4(\lambda_{\sigma(p+1)},\ldots,\lambda_{\sigma(n)};\mu_{\sigma'(p+1)},\ldots,\mu_{\sigma'(n)})\Big\}. \quad (4.52)$$

**Remark:** In the derivation of (4.52), the spectral parameters $\{\lambda_i\}$ in the state $|\Omega_N(\{\lambda_j\}_{(p,n)})\rangle$ are required to satisfy the BAE (2.22). However, the parameters $\mu_j$ $(j=1,\ldots,n)$ in the dual state $\langle\Omega_N(\{\mu_j\}_{(p,n)})|$ do not need to satisfy the BAE.

On the other hand, if we compute the scalar product by starting from the dual state $\langle\Omega_N(\{\lambda_j\}_{(p,n)})|$, then by using the same procedure, we have

$$G^{(n)}(\{\lambda_k\}_{(p,n)},\{\mu_j\}_{(p,n)}) = G^{(n)}(\{\mu_j\}_{(p,n)},\{\lambda_k\}_{(p,n)}). \quad (4.53)$$

28

Therefore, the corresponding scalar product $P_n^L(\{\lambda_k\}_{(p,n)}, \{\mu_j\}_{(p,n)})$ is given by

$$
\begin{aligned}
&P_n^L(\{\lambda_k\}_{(p,n)}, \{\mu_j\}_{(p,n)}) \\
&= \sum_{\sigma,\sigma' \in \mathcal{S}_n} Y_L(\{\lambda_{\sigma(j)}\}, \{\lambda_{\sigma(k)}^{(1)}\}) Y_R(\{\mu_{\sigma'(j)}\} \{\mu_{\sigma'(k)}^{(1)}\}) \ G^{(n)}(\{\mu_{\sigma'(j)}\}_{(p,n)}, \{\lambda_{\sigma(k)}\}_{(p,n)}) (4.54)
\end{aligned}
$$

In (4.54), we have also assumed that any element of the spectral parameter set $\{\lambda_i\}$ satisfy the BAE.

## 5   Correlation functions

Having obtained the scalar product and the norm, we are now in the position to compute the k-point correlation functions of the model. In general, a k-point correlation function is defined by

$$
F_n^{\epsilon^1,\ldots,\epsilon^k} = \langle \Omega_N(\{\mu_j\})|\epsilon_{i_1}^1 \ldots \epsilon_{i_k}^k|\Omega_N(\{\lambda_j\})\rangle, \tag{5.1}
$$

where $\epsilon_{i_j}^j$ stand for the local fermion representations, (2.14), of the generators of the super-algebra $gl(2|1)$, and the lower indices $i_j$ indicate the positions of the fermion operators.

The authors in [28] proved that the local spin and field operators of the fundamental graded models can be represented in terms of monodromy matrix. Specializing to the current system, we obtain

$$
(1-n_{\kappa,\uparrow})c_{\kappa,\downarrow} = \prod_{j=1}^{\kappa-1} t(\xi_j) \cdot B_1(\xi_\kappa) \cdot \prod_{j=\kappa+1}^{N} t(\xi_j) \quad, \tag{5.2}
$$

$$
(1-n_{\kappa,\downarrow})c_{\kappa,\uparrow} = \prod_{j=1}^{\kappa-1} t(\xi_j) \cdot B_2(\xi_\kappa) \cdot \prod_{j=\kappa+1}^{N} t(\xi_j) \quad, \tag{5.3}
$$

$$
(1-n_{\kappa,\uparrow})c_{\kappa,\downarrow}^\dagger = \prod_{j=1}^{\kappa-1} t(\xi_j) \cdot C_1(\xi_\kappa) \cdot \prod_{j=\kappa+1}^{N} t(\xi_j) \quad, \tag{5.4}
$$

$$
(1-n_{\kappa,\downarrow})c_{\kappa,\uparrow}^\dagger = \prod_{j=1}^{n-1} t(\xi_j) \cdot C_2(\xi_\kappa) \cdot \prod_{j=\kappa+1}^{N} t(\xi_j) \quad, \tag{5.5}
$$

$$
(1-n_{\kappa,\downarrow})(1-n_{\kappa,\uparrow}) = \prod_{j=1}^{\kappa-1} t(\xi_j) \cdot D(\xi_\kappa) \cdot \prod_{j=\kappa+1}^{N} t(\xi_j) \quad, \tag{5.6}
$$

$$
S_\kappa^\dagger = -\prod_{j=1}^{\kappa-1} t(\xi_j) \cdot A_{21}(\xi_\kappa) \cdot \prod_{j=\kappa+1}^{N} t(\xi_j) \quad, \tag{5.7}
$$

29

$$S_\kappa = -\prod_{j=1}^{\kappa-1} t(\xi_j) \cdot A_{12}(\xi_\kappa) \cdot \prod_{j=\kappa+1}^{N} t(\xi_j) \quad , \tag{5.8}$$

$$S_\kappa^z = -\frac{1}{2}\prod_{j=1}^{\kappa-1} t(\xi_j) \cdot (A_{11}(\xi_\kappa) - A_{22}(\xi_\kappa)) \cdot \prod_{j=\kappa+1}^{N} t(\xi_j) \quad , \tag{5.9}$$

where in the left of (5.6), $n_{\kappa\downarrow}n_{\kappa\uparrow} = 0$ since the double occupancy of lattice sites on the restriced Hilbert space of the supersymmetric $t$-$J$ model is excluded.

## 5.1   One-point functions

We first calculate one point correlation functions for the local fermion operators $(1-n_{\kappa,\downarrow})c_{\kappa,\uparrow}$ and $(1-n_{\kappa,\downarrow})c_{\kappa,\uparrow}^\dagger$. According to (5.1), the correlation functions are given by

$$
\begin{aligned}
&F_n^\uparrow(\{\mu_j\}_{(p,n)}, \xi_\kappa, \{\lambda_k\}_{(p+1,n+1)}) \\
&= \langle\Omega_N(\{\mu_j\}_{(p,n)})| \cdot (1-n_{\kappa,\downarrow})c_{\kappa,\uparrow} \cdot |\Omega_N(\{\lambda_j\}_{(p+1,n+1)})\rangle \\
&= \sum_{\sigma\in S_{n+1}}\sum_{\sigma'\in S_n} Y_L(\{\mu_{\sigma'(j)}\}, \{\mu_{\sigma'(k)}^{(1)}\})Y_R(\{\lambda_{\sigma(j)}\}, \{\lambda_{\sigma(k)}^{(1)}\}) \\
&\quad \times\langle 0|\overset{\overleftarrow{n}}{\prod_{i=p+1}} B_1(\mu_{\sigma'(i)})\overset{\overleftarrow{p}}{\prod_{i=1}}B_2(\mu_{\sigma'(i)}) \cdot (1-n_{\kappa,\downarrow})c_{\kappa,\uparrow}^\dagger \cdot \overset{\overrightarrow{p+1}}{\prod_{i=1}}C_2(\lambda_{\sigma(i)})\overset{\overrightarrow{n+1}}{\prod_{i=p+2}}C_1(\lambda_{\sigma(i)})|0\rangle,
\end{aligned}
$$
$$\tag{5.10}$$

$$
\begin{aligned}
&F_n^{\uparrow\dagger}(\{\lambda_j\}_{(p+1,n+1)}, \xi_\kappa, \{\mu_k\}_{(p,n)}) \\
&= \langle\Omega_N(\{\lambda_j\}_{(p+1,n+1)})| \cdot (1-n_{\kappa,\downarrow})c_{\kappa,\uparrow}^\dagger \cdot |\Omega_N(\{\mu_j\}_{(p,n)})\rangle \\
&= \sum_{\sigma'\in S_n}\sum_{\sigma\in S_{n+1}} Y_L(\{\lambda_{\sigma(j)}\}, \{\lambda_{\sigma(k)}^{(1)}\})Y_R(\{\mu_{\sigma'(j)}\}, \{\mu_{\sigma'(k)}^{(1)}\}) \\
&\quad \times\langle 0|\overset{\overleftarrow{n+1}}{\prod_{i=p+2}} B_1(\lambda_{\sigma(i)})\overset{\overleftarrow{p+1}}{\prod_{i=1}}B_2(\lambda_{\sigma(i)}) \cdot (1-n_{\kappa,\downarrow})c_{\kappa,\uparrow}^\dagger \cdot \overset{\overrightarrow{p}}{\prod_{i=1}}C_2(\mu_{\sigma'(i)})\overset{\overrightarrow{n}}{\prod_{i=p+1}}C_1(\mu_{\sigma'(i)})|0\rangle,
\end{aligned}
$$
$$\tag{5.11}$$

where $\{\mu_j\}, \{\lambda_k\}$ are solutions of BAE, $p$ and $p+1$ are quantum numbers of the corresponding states. For the representations of the correlation functions, we prove the following proposition:

**Proposition 3** *If both the Bethe state $|\Omega_N(\{u_j\})\rangle$ and the dual Bethe state $\langle\Omega_N(\{u_j\})|$ ($u_j = \lambda_j, \mu_j$) are eigenstates of the transfer matrix, then the correlation functions corresponding to*

the local fermion operators $(1 - n_{\kappa,\downarrow})c_{\kappa,\uparrow}$ and $(1 - n_{\kappa,\downarrow})c_{\kappa,\uparrow}^\dagger$ can be represented by

$$F_n^\uparrow(\{\mu_j\}_{(p,n)}, \xi_\kappa, \{\lambda_k\}_{(p+1,n+1)})$$

$$= (-1)^{n+1} \sum_{\sigma \in \mathcal{S}_{n+1}} \sum_{\sigma' \in \mathcal{S}_n} \phi_{\kappa-1}(\{\mu_j\}) \phi_\kappa^{-1}(\{\lambda_k\})$$

$$\times P_{n+1}\left(\xi_\kappa, \mu_{\sigma'(1)}, \ldots, \mu_{\sigma'(n)}; \{\lambda_{\sigma(j)}\}_{(p+1,n+1)}\right), \tag{5.12}$$

$$F_n^{\uparrow\dagger}(\{\lambda_j\}_{(p+1,n+1)}, \kappa, \{\mu_k\}_{(p,n)})$$

$$= (-1)^{n+1} \sum_{\sigma \in \mathcal{S}_{n+1}} \sum_{\sigma' \in \mathcal{S}_n} \phi_{\kappa-1}(\{\lambda_j\}) \phi_\kappa^{-1}(\{\mu_k\})$$

$$\times P_{n+1}^L\left(\{\lambda_{\sigma(j)}\}_{(p+1,n+1)}; \xi_\kappa, \mu_{\sigma'(1)}, \ldots, \mu_{\sigma'(n)}\right), \tag{5.13}$$

respectively, where $\phi_i(\{\mu_j\}) = \prod_{k=1}^i \prod_{l=1}^n a^{-1}(\mu_l, \xi_k)$.

*Proof.* We first prove (5.12). From the definition of $F_n^\uparrow$, we have

$$F_{n+1}^\uparrow(\{\mu_j\}_{(p,n)}, \xi_\kappa, \{\lambda_k\}_{(p+1,n+1)})$$

$$= \langle\Omega_N(\{\mu_j\}_{(p,n)})| \cdot (1 - n_{\kappa,\downarrow})c_{\kappa,\uparrow} \cdot |\Omega_N(\{\lambda_j\}_{(p+1,n+1)})\rangle$$

$$= \langle\Omega_N(\{\mu_j\}_{(p,n)})| \prod_{j=1}^{\kappa-1} t(\xi_j) \cdot B_2(\xi_\kappa) \cdot \prod_{j=\kappa+1}^N t(\xi_j) |\Omega_N(\{\lambda_j\}_{(p+1,n+1)})\rangle$$

$$= (-1)^{n+1} \sum_{\sigma \in \mathcal{S}_{n+1}} \sum_{\sigma' \in \mathcal{S}_n} \prod_{j=1}^n \prod_{k=1}^{\kappa-1} a^{-1}(\mu_j, \xi_k) \prod_{j=1}^{n+1} \prod_{k=\kappa+1}^N a^{-1}(\lambda_j, \xi_k)$$

$$\times P_{n+1}\left(\xi_\kappa, \mu_{\sigma'(1)}, \ldots, \mu_{\sigma'(n)}; \{\lambda_{\sigma(j)}\}_{(p+1,n+1)}\right). \tag{5.14}$$

Then by using the relation

$$\prod_{j=1}^n \prod_{k=1}^N a^{-1}(\lambda_j, \xi_k) = 1, \tag{5.15}$$

which is from the BAE and the NBAE, we prove (5.12). The proof of (5.13) is similar.

$\square$

For the local operators $(1 - n_{\kappa,\uparrow})c_{\kappa,\downarrow}$ and $(1 - n_{\kappa\uparrow})c_{\kappa\downarrow}^\dagger$, the calculation of their correlation functions, which are defined by

$$F_{n+1}^\downarrow(\{\mu_j\}_{(p,n)}, \xi_\kappa, \{\lambda_k\}_{(p,n+1)}) = \langle\Omega_N(\{\mu_j\}_{(p,n)})| \cdot (1 - n_{\kappa,\uparrow})c_{\kappa,\downarrow} \cdot |\Omega_N(\{\lambda_k\}_{(p,n+1)})\rangle, \tag{5.16}$$

$$F_{n+1}^{\downarrow\dagger}(\{\lambda_j\}_{(p,n+1)}, \xi_\kappa, \{\mu_k\}_{(p,n)}) = \langle\tilde{\Omega}_N(\{\lambda_k\}_{(p,n+1)})| \cdot (1 - n_{\kappa,\uparrow})c_{\kappa,\downarrow}^\dagger \cdot |\Omega_N(\{\mu_j\}_{(p,n)})\rangle, \tag{5.17}$$

respectively, leads to the following proposition:

**Proposition 4** *If both the Bethe state $|\Omega_N(\{u_j\})\rangle$ and the dual Bethe state $\langle\Omega_N(\{u_j\})|$ $(u_j = \lambda_j, \mu_j)$ are eigenstates of the transfer matrix, then the correlation functions corresponding to the local fermion operators $(1 - n_{\kappa,\uparrow})c_{\kappa,\downarrow}$ and $(1 - n_{\kappa,\uparrow})c^{\dagger}_{\kappa,\downarrow}$ can be represented by*

$$
F^{\downarrow}_{n+1}(\{\mu_j\}_{(p,n)}, \xi_\kappa, \{\lambda_k\}_{(p,n+1)})
$$
$$
= (-1)^{n+1} \sum_{\sigma\in\mathcal{S}_{n+1}} \sum_{\sigma'\in\mathcal{S}_n} \phi_{\kappa-1}(\{\mu_j\})\phi_\kappa^{-1}(\{\lambda_k\})
$$
$$
\times \Bigg\{ (-1)^p \prod_{j=1}^{p} \frac{1}{a(\mu_{\sigma'(j)}, \xi_\kappa)} P_{n+1}\left(\mu_{\sigma'(1)}, \ldots, \mu_{\sigma'(p)}, \xi_\kappa, \mu_{\sigma'(p+1)}, \ldots, \mu_{\sigma'(n)}; \{\lambda_{\sigma(j)}\}_{(p,n+1)}\right)
$$
$$
+ (-1)^{p-1} \sum_{j=1}^{p} \frac{b(\mu_{\sigma'(j)}, \xi_\kappa)}{a(\mu_{\sigma'(j)}, \xi_\kappa)} \prod_{k=1}^{j-1} \frac{c(\mu_{\sigma'(k)}, \mu_{\sigma'(j)})}{c(\mu_{\sigma'(k)}, \xi_\kappa)} \prod_{l=1,\neq j}^{p} \frac{1}{a(\mu_{\sigma'(l)}, \mu_{\sigma'(j)})}
$$
$$
\times P_{n+1}\left(\mu_{\sigma'(1)}, \ldots, \mu_{\sigma'(j-1)}, \xi_\kappa, \mu_{\sigma'(j+1)}, \ldots, \mu_{\sigma'(p)}, \right.
$$
$$
\left. \mu_{\sigma'(j)}, \mu_{\sigma'(p+1)}, \ldots, \mu_{\sigma'(n)}; \{\lambda_{\sigma(j)}\}_{(p,n+1)}\right) \Bigg\},
$$

(5.18)

$$
F^{\downarrow\dagger}_{n+1}(\{\lambda_j\}_{(p,n+1)}, \xi_\kappa, \{\mu_k\}_{(p,n)})
$$
$$
= (-1)^{n+1} \sum_{\sigma\in\mathcal{S}_{n+1}} \sum_{\sigma'\in\mathcal{S}_n} \phi_{\kappa-1}(\{\lambda_j\})\phi_\kappa^{-1}(\{\mu_k\})
$$
$$
\times \Bigg\{ (-1)^p \prod_{j=1}^{p} \frac{1}{a(\mu_{\sigma'(j)}, \xi_\kappa)} P^L_{n+1}\left(\{\lambda_{\sigma(j)}\}_{(p+1,n+1)}; \mu_{\sigma'(1)}, \ldots, \mu_{\sigma'(p)}, \xi_\kappa, \mu_{\sigma'(p+1)}, \ldots, \mu_{\sigma'(n)}\right)
$$
$$
+ (-1)^{p-1} \sum_{j=1}^{p} \frac{b(\mu_{\sigma'(j)}, \xi_\kappa)}{a(\mu_{\sigma'(j)}, \xi_\kappa)} \prod_{k=1}^{j-1} \frac{c(\mu_{\sigma'(k)}, \mu_{\sigma'(j)})}{c(\mu_{\sigma'(k)}, \xi_\kappa)} \prod_{l=1,\neq j}^{p} \frac{1}{a(\mu_{\sigma'(l)}, \mu_{\sigma'(j)})}
$$
$$
\times P^L_{n+1}\left(\{\lambda_{\sigma(j)}\}_{(p,n+1)}; \mu_{\sigma'(1)}, \ldots, \mu_{\sigma'(j-1)}, \xi_\kappa, \mu_{\sigma'(j+1)}, \ldots, \mu_{\sigma'(p)}, \right.
$$
$$
\left. \mu_{\sigma'(j)}, \mu_{\sigma'(p+1)}, \ldots, \mu_{\sigma'(n)}\right) \Bigg\},
$$

(5.19)

*respectively.*

*Proof.* We first prove (5.18). Considering the definition of $F^{\downarrow}_n$ the representation of the local fermion operator (5.3), we have

$$
F^{\downarrow}_{n+1}(\{\mu_j\}_{(p,n)}, \xi_\kappa, \{\lambda_k\}_{(p,n+1)})
$$
$$
= \langle\Omega_N(\{\mu_j\}_{(p,n)})| \cdot \prod_{j=1}^{\kappa-1} t(\xi_j) \cdot B_1(\xi_\kappa) \cdot \prod_{j=\kappa+1}^{N} t(\xi_j)|\Omega_N(\{\lambda_k\}_{(p,n+1)})\rangle,
$$

$$= \sum_{\sigma \in \mathcal{S}_{n+1}} \sum_{\sigma' \in \mathcal{S}_n} \phi_{\kappa-1}(\{\lambda_{\sigma(j)}\}) \phi_\kappa^{-1}(\{\mu_{\sigma'(k)}\}) Y_L(\{\mu_{\sigma'(j)}\}, \{\mu_{\sigma'(k)}^{(1)}\}) Y_R(\{\lambda_{\sigma(j)}\}, \{\lambda_{\sigma(k)}^{(1)}\})$$

$$\times \langle 0 | \overset{\overset{\longleftarrow}{n}}{\prod_{i=p+1}} B_1(\mu_{\sigma'(i)}) \overset{\overset{\longleftarrow}{p}}{\prod_{i=1}} B_2(\mu_{\sigma'(i)}) \cdot B_1(\xi_\kappa) \cdot \overset{\overset{\longrightarrow}{p+1}}{\prod_{i=1}} C_2(\lambda_{\sigma(i)}) \overset{\overset{\longrightarrow}{n+1}}{\prod_{i=p+2}} C_1(\lambda_{\sigma(i)}) | 0 \rangle. \tag{5.20}$$

From the GYBE (2.9), we have the following commutation relations

$$B_a(\mu) B_a(\lambda) = -c(\lambda, \mu) B_a(\lambda) B_a(\mu) \tag{5.21}$$

$$B_a(\mu) B_b(\lambda) = -\frac{1}{a(\mu, \lambda)} B_b(\lambda) B_a(\mu) + \frac{b(\mu, \lambda)}{a(\mu, \lambda)} B_b(\mu) B_a(\lambda) \quad (a \neq b). \tag{5.22}$$

Then we have

$$B_2(\mu_p) \dots B_2(\mu_1) B_1(\xi_\kappa)$$

$$= (-1)^p \prod_{j=1}^{p} \frac{1}{a(\mu_j, \xi_\kappa)} B_1(\xi_\kappa) B_2(\mu_p) \dots B_2(\mu_1)$$

$$+ (-1)^{p-1} \sum_{j=1}^{p} \frac{b(\mu_j, \xi_\kappa)}{a(\mu_j, \xi_\kappa)} \prod_{k=1}^{j-1} \frac{c(\mu_k, \mu_j)}{c(\mu_k, \xi_\kappa)} \prod_{l=1, \neq j}^{p} \frac{1}{a(\mu_l, \mu_j)}$$

$$\times B_1(\mu_j) B_2(\mu_p) \dots B_2(\mu_{j+1}) B_2(\xi_\kappa) B_2(\mu_{j-1}) \dots B_2(\mu_1). \tag{5.23}$$

Substituting the about relation into (5.20), we obtain (5.18).

Similarly, by using the commutation relations

$$C_a(\lambda) C_a(\mu) = -c(\lambda, \mu) C_a(\mu) C_a(\lambda) \tag{5.24}$$

$$C_a(\lambda) C_b(\mu) = -\frac{1}{a(\mu, \lambda)} C_b(\mu) C_a(\lambda) + \frac{b(\mu, \lambda)}{a(\mu, \lambda)} C_b(\lambda) C_a(\mu) \quad (a \neq b). \tag{5.25}$$

one may prove (5.19).

$\square$

For correlation functions associated with the local fermion operators $S_\kappa$ and $S_\kappa^\dagger$ defined by

$$F_n^S(\{\mu_j\}_{(p,n)}, \xi_\kappa, \{\lambda_k\}_{(p-1,n)}) = \langle \Omega_N(\{\mu_j\}_{(p,n)})| \cdot S \cdot |\Omega_N(\{\lambda_k\}_{(p-1,n)})\rangle, \tag{5.26}$$

$$F_n^{S^\dagger}(\{\lambda_j\}_{(p-1,n)}, \xi_\kappa, \{\mu_k\}_{(p,n)}) = \langle \Omega_N(\{\lambda_k\}_{(p-1,n)})| \cdot S^\dagger \cdot |\Omega_N(\{\mu_j\}_{(p,n)})\rangle, \tag{5.27}$$

we have the following proposition:

**Proposition 5** *If both the Bethe state $|\Omega_N(\{u_j\})\rangle$ and the dual Bethe state $\langle\Omega_N(\{u_j\})|$ ($u_j = \lambda_j, \mu_j$) are eigenstates of the transfer matrix, then the correlation functions corresponding to the local fermion operators $S$ and $S^\dagger$ can be represented by*

$$
F_n^S(\{\mu_j\}_{(p,n)}, \xi_\kappa, \{\lambda_k\}_{(p-1,n)})
$$

$$
= (-1)^{n+1} \sum_{\sigma \in \mathcal{S}_n} \sum_{\sigma' \in \mathcal{S}_n} \phi_{\kappa-1}(\{\mu_j\}) \phi_\kappa^{-1}(\{\lambda_k\}) \sum_{i=1}^p \frac{b(\xi_\kappa, \mu_{\sigma'(i)})}{a(\xi_\kappa, \mu_{\sigma'(i)})}
$$

$$
\times \Bigg\{ (-1)^{p-i} \prod_{j=i+1}^p \frac{c(\mu_{\sigma'(i)}, \mu_{\sigma'(j)})}{a(\mu_{\sigma'(i)}, \mu_{\sigma'(j)})} \prod_{k=i+1}^p \frac{1}{a(\mu_{\sigma'(k)}, \xi_\kappa)} \prod_{\alpha=1}^N a(\mu_{\sigma'(i)}, \xi_\alpha)
$$

$$
\times P_n \Big( \{\mu_{\sigma'(1)}, \ldots, \mu_{\sigma'(i-1)}, \mu_{\sigma'(i+1)}, \ldots, \mu_{\sigma'(p)}, \xi_\kappa, \mu_{\sigma'(p+1)}, \ldots, \mu_{\sigma'(n)}\}_{(p-1,n)};
$$

$$
\{\lambda_{\sigma(j)}\}_{(p-1,n)} \Big)
$$

$$
+ (-1)^{p-i-1} \prod_{j=i+1}^p \frac{c(\mu_{\sigma'(i)}, \mu_{\sigma'(j)})}{a(\mu_{\sigma'(i)}, \mu_{\sigma'(j)})} \sum_{k=i+1}^p \frac{b(\mu_{\sigma'(k)}, \xi_\kappa)}{a(\mu_{\sigma'(k)}, \xi_\kappa)} \prod_{l=i+1}^{k-1} \frac{c(\mu_{\sigma'(l)}, \mu_{\sigma'(k)})}{c(\mu_{\sigma'(l)}, \xi_\kappa)}
$$

$$
\times \prod_{m=i+1, \neq k}^p \frac{1}{a(\mu_{\sigma'(m)}, \mu_{\sigma'(k)})} \prod_{\alpha=1}^N a(\mu_{\sigma'(i)}, \xi_\alpha) \, P_n \Big( \{\mu_{\sigma'(1)}, \ldots, \mu_{\sigma'(i-1)}, \mu_{\sigma'(i+1)}, \ldots
$$

$$
\mu_{\sigma'(k-1)}, \xi_\kappa, \mu_{\sigma'(k+1)}, \ldots, \mu_{\sigma'(p)}, \mu_{\sigma'(k)}, \mu_{\sigma'(p+1)}, \ldots, \mu_{\sigma'(n)}\}_{(p-1,n)}; \{\lambda_{\sigma(j)}\}_{(p-1,n)} \Big),
$$

$$
+ \sum_{j=i+1}^p \frac{b(\mu_{\sigma'(i)}, \mu_{\sigma'(j)})}{a(\mu_{\sigma'(i)}, \mu_{\sigma'(j)})} \prod_{k=i+1}^{j-1} \frac{c(\mu_{\sigma'(k)}, \mu_{\sigma'(j)})}{c(\mu_{\sigma'(k)}, \mu_{\sigma'(i)})} \prod_{l=i+1, \neq j}^p \frac{c(\mu_{\sigma'(j)}, \mu_{\sigma'(l)})}{a(\mu_{\sigma'(j)}, \mu_{\sigma'(l)})} \prod_{\alpha=1}^N a(\mu_{\sigma'(j)}, \xi_\alpha)
$$

$$
\times \Bigg[ (-1)^{p-i} \prod_{m=i+1}^p \frac{1}{a(\mu'_{\sigma'(m)}, \xi_\kappa)} \, P_n \Big( \{\mu_{\sigma'(1)}, \ldots, \mu_{\sigma'(i-1)}, \mu'_{\sigma'(i+1)}, \ldots, \mu'_{\sigma'(p)},
$$

$$
\xi_\kappa, \mu_{\sigma'(p+1)}, \ldots, \mu_{\sigma'(n)}\}_{(p-1,n)}; \{\lambda_{\sigma(j)}\}_{(p-1,n)} \Big)
$$

$$
+ (-1)^{p-i-1} \sum_{m=i+1}^p \frac{b(\mu'_{\sigma'(m)}, \xi_\kappa)}{a(\mu'_{\sigma'(m)}, \xi_\kappa)} \prod_{n=i+1}^{m-1} \frac{c(\mu'_{\sigma'(n)}, \mu'_{\sigma'(m)})}{c(\mu'_{\sigma'(n)}, \xi_\kappa)} \prod_{s=i+1, \neq m}^p \frac{1}{a(\mu'_{\sigma'(s)}, \mu'_{\sigma'(m)})}
$$

$$
\times P_n \Big( \{\mu_{\sigma'(1)}, \ldots, \mu_{\sigma'(i-1)}, \mu'_{\sigma'(i+1)}, \ldots \mu'_{\sigma'(m-1)}, \xi_\kappa, \mu'_{\sigma'(m+1)}, \ldots, \mu'_{\sigma'(p)}, \mu'_{\sigma'(m)},
$$

$$
\mu_{\sigma'(p+1)}, \ldots, \mu_{\sigma'(n)}\}_{(p-1,n)}; \{\lambda_{\sigma(j)}\}_{(p-1,n)} \Big) \Bigg]
$$

$$
- \sum_{j=i+1}^p \frac{b(\xi_\kappa, \mu_{\sigma'(j)})}{a(\xi_\kappa, \mu_{\sigma'(j)})} \prod_{k=i+1}^{j-1} \frac{c(\mu_{\sigma'(k)}, \mu_{\sigma'(j)})}{c(\mu_{\sigma'(k)}, \xi_\kappa)} \prod_{l=i+1, \neq j}^p \frac{c(\mu_{\sigma'(j)}, \mu_{\sigma'(l)})}{a(\mu_{\sigma'(j)}, \mu_{\sigma'(l)})} \prod_{\alpha=1}^N a(\mu_{\sigma'(j)}, \xi_\alpha)
$$

$$
\times \Bigg[ (-1)^{p-i} \prod_{m=i+1}^p \frac{1}{a(\mu''_{\sigma'(m)}, \mu_{\sigma'(i)})} \, P_n \Big( \{\mu_{\sigma'(1)}, \ldots, \mu_{\sigma'(i-1)}, \mu''_{\sigma'(i+1)}, \ldots, \mu''_{\sigma'(p)},
$$

$$
\mu_{\sigma'(i)}, \mu_{\sigma'(p+1)}, \ldots, \mu_{\sigma'(n)}\}_{(p-1,n)}; \{\lambda_{\sigma(j)}\}_{(p-1,n)} \Big)
$$

$$
+ (-1)^{p-i-1} \sum_{m=i+1}^p \frac{b(\mu''_{\sigma'(m)}, \mu_{\sigma'(i)})}{a(\mu''_{\sigma'(m)}, \mu_{\sigma'(i)})} \prod_{n=i+1}^{m-1} \frac{c(\mu''_{\sigma'(n)}, \mu''_{\sigma'(m)})}{c(\mu''_{\sigma'(n)}, \mu_{\sigma'(i)})} \prod_{s=i+1, \neq m}^p \frac{1}{a(\mu''_{\sigma'(s)}, \mu''_{\sigma'(m)})}
$$

$$\times P_n\left(\{\mu_{\sigma'(1)},\ldots,\mu_{\sigma'(i-1)},\mu''_{\sigma'(i+1)},\ldots\mu''_{\sigma'(m-1)},\mu_{\sigma'(i)},\mu''_{\sigma'(m+1)},\ldots,\mu''_{\sigma'(p)},\mu''_{\sigma'(m)},\right.$$

$$\left.\mu_{\sigma'(p+1)},\ldots,\mu_{\sigma'(n)}\}_{(p-1,n)};\{\lambda_{\sigma(j)}\}_{(p-1,n)})\Big]\Big\}, \tag{5.28}$$

$$F_n^{S^\dagger}(\{\lambda_k\}_{(p-1,n)},\xi_\kappa,\{\mu_j\}_{(p,n)})$$

$$= (-1)^{n+1}\sum_{\sigma\in\mathcal{S}_n}\sum_{\sigma'\in\mathcal{S}_n}\phi_{\kappa-1}(\{\lambda_j\})\phi_\kappa^{-1}(\{\mu_k\})\sum_{i=1}^p\frac{b(\xi_\kappa,\mu_{\sigma'(i)})}{a(\xi_\kappa,\mu_{\sigma'(i)})}$$

$$\times\left\{(-1)^{p-i}\prod_{j=i+1}^p\frac{c(\mu_{\sigma'(i)},\mu_{\sigma'(j)})}{a(\mu_{\sigma'(i)},\mu_{\sigma'(j)})}\prod_{k=i+1}^p\frac{1}{a(\mu_{\sigma'(k)},\xi_\kappa)}\prod_{\alpha=1}^N a(\mu_{\sigma'(i)},\xi_\alpha)\right.$$

$$\times P_n^L\left(\{\lambda_{\sigma(d)}\}_{(p-1,n)},\{\mu_{\sigma'(1)},\ldots,\mu_{\sigma'(i-1)},\mu_{\sigma'(i+1)},\ldots,\mu_{\sigma'(p)},\xi_\kappa,\right.$$

$$\left.\mu_{\sigma'(p+1)},\ldots,\mu_{\sigma'(n)}\}_{(p-1,n)}\right)$$

$$+ (-1)^{p-i-1}\prod_{j=i+1}^p\frac{c(\mu_{\sigma'(i)},\mu_{\sigma'(j)})}{a(\mu_{\sigma'(i)},\mu_{\sigma'(j)})}\sum_{k=i+1}^p\frac{b(\mu_{\sigma'(k)},\xi_\kappa)}{a(\mu_{\sigma'(k)},\xi_\kappa)}\prod_{l=i+1}^{k-1}\frac{c(\mu_{\sigma'(l)},\mu_{\sigma'(k)})}{c(\mu_{\sigma'(l)},\xi_\kappa)}$$

$$\times\prod_{m=i+1,\neq k}^p\frac{1}{a(\mu_{\sigma'(m)},\mu_{\sigma'(k)})}\prod_{\alpha=1}^N a(\mu_{\sigma'(i)},\xi_\alpha)\,P_n^L\left(\{\lambda_{\sigma(d)}\}_{(p-1,n)},\{\mu_{\sigma'(1)},\ldots,\mu_{\sigma'(i-1)},\right.$$

$$\left.\mu_{\sigma'(i+1)},\ldots,\mu_{\sigma'(k-1)},\xi_\kappa,\mu_{\sigma'(k+1)},\ldots,\mu_{\sigma'(p)},\mu_{\sigma'(k)},\mu_{\sigma'(p+1)},\ldots,\mu_{\sigma'(n)}\}_{(p-1,n)}\right),$$

$$+ \sum_{j=i+1}^p\frac{b(\mu_{\sigma'(i)},\mu_{\sigma'(j)})}{a(\mu_{\sigma'(i)},\mu_{\sigma'(j)})}\prod_{k=i+1}^{j-1}\frac{c(\mu_{\sigma'(k)},\mu_{\sigma'(j)})}{c(\mu_{\sigma'(k)},\mu_{\sigma'(i)})}\prod_{l=i+1,\neq j}^p\frac{c(\mu_{\sigma'(j)},\mu_{\sigma'(l)})}{a(\mu_{\sigma'(j)},\mu_{\sigma'(l)})}\prod_{\alpha=1}^N a(\mu_{\sigma'(j)},\xi_\alpha)$$

$$\times\left[(-1)^{p-i}\prod_{m=i+1}^p\frac{1}{a(\mu'_{\sigma'(m)},\xi_\kappa)}\,P_n^L\left(\{\lambda_{\sigma(d)}\}_{(p-1,n)};\{\mu_{\sigma'(1)},\ldots,\mu_{\sigma'(i-1)},\right.\right.$$

$$\left.\mu'_{\sigma'(i+1)},\ldots,\mu'_{\sigma'(p)},\xi_\kappa,\mu_{\sigma'(p+1)},\ldots,\mu_{\sigma'(n)}\}_{(p-1,n)}\right)$$

$$+ (-1)^{p-i-1}\sum_{m=i+1}^p\frac{b(\mu'_{\sigma'(m)},\xi_\kappa)}{a(\mu'_{\sigma'(m)},\xi_\kappa)}\prod_{n=i+1}^{m-1}\frac{c(\mu'_{\sigma'(n)},\mu'_{\sigma'(m)})}{c(\mu'_{\sigma'(n)},\xi_\kappa)}\prod_{s=i+1,\neq m}^p\frac{1}{a(\mu'_{\sigma'(s)},\mu'_{\sigma'(m)})}$$

$$\times P_n^L\left(\{\lambda_{\sigma(d)}\}_{(p-1,n)};\{\mu_{\sigma'(1)},\ldots,\mu_{\sigma'(i-1)},\mu'_{\sigma'(i+1)},\ldots\mu'_{\sigma'(m-1)},\xi_\kappa,\mu'_{\sigma'(m+1)},\ldots,\right.$$

$$\left.\left.\mu'_{\sigma'(p)},\mu'_{\sigma'(m)},\mu_{\sigma'(p+1)},\ldots,\mu_{\sigma'(n)}\}_{(p-1,n)}\right)\right]$$

$$- \sum_{j=i+1}^p\frac{b(\xi_\kappa,\mu_{\sigma'(j)})}{a(\xi_\kappa,\mu_{\sigma'(j)})}\prod_{k=i+1}^{j-1}\frac{c(\mu_{\sigma'(k)},\mu_{\sigma'(j)})}{c(\mu_{\sigma'(k)},\xi_\kappa)}\prod_{l=i+1,\neq j}^p\frac{c(\mu_{\sigma'(j)},\mu_{\sigma'(l)})}{a(\mu_{\sigma'(j)},\mu_{\sigma'(l)})}\prod_{\alpha=1}^N a(\mu_{\sigma'(j)},\xi_\alpha)$$

$$\times\left[(-1)^{p-i}\prod_{m=i+1}^p\frac{1}{a(\mu''_{\sigma'(m)},\mu_{\sigma'(i)})}\,P_n^L\left(\{\lambda_{\sigma(j)}\}_{(p-1,n)};\{\mu_{\sigma'(1)},\ldots,\mu_{\sigma'(i-1)},\right.\right.$$

$$\left.\mu''_{\sigma'(i+1)},\ldots,\mu''_{\sigma'(p)},\mu_{\sigma'(i)},\mu_{\sigma'(p+1)},\ldots,\mu_{\sigma'(n)}\}_{(p-1,n)}\right)$$

$$+ (-1)^{p-i-1}\sum_{m=i+1}^p\frac{b(\mu''_{\sigma'(m)},\mu_{\sigma'(i)})}{a(\mu''_{\sigma'(m)},\mu_{\sigma'(i)})}\prod_{n=i+1}^{m-1}\frac{c(\mu''_{\sigma'(n)},\mu''_{\sigma'(m)})}{c(\mu''_{\sigma'(n)},\mu_{\sigma'(i)})}\prod_{s=i+1,\neq m}^p\frac{1}{a(\mu''_{\sigma'(s)},\mu''_{\sigma'(m)})}$$

$$\times P_n^L\left(\{\lambda_{\sigma(d)}\}_{(p-1,n)};\{\mu_{\sigma'(1)},\ldots,\mu_{\sigma'(i-1)},\mu''_{\sigma'(i+1)},\ldots\mu''_{\sigma'(m-1)},\mu_{\sigma'(i)},\right.$$

$$\mu''_{\sigma'(m+1)}, \ldots, \mu''_{\sigma'(p)}, \mu''_{\sigma'(m)}, \mu_{\sigma'(p+1)}, \ldots, \mu_{\sigma'(n)}\}_{(p-1,n)}\big)\big]\Big\}\,, \tag{5.29}$$

*respectively, where $\mu'_{\sigma'(k)} = \mu_{\sigma'(k)}$, for $k = i+1, \ldots, j-1, j+1, \ldots, p$, $\mu'_{\sigma'(k)} = \mu_{\sigma'(i)}$, for $k = j$, $\mu''_{\sigma'(k)} = \mu_{\sigma'(k)}$, for $k = i+1, \ldots, j-1, j+1, \ldots, p$ and $\mu''_{\sigma'(k)} = \xi_\kappa$, for $k = j$.*

In proving this proposition, we have used the commutation relations

$$A_{ab}(\lambda)C_c(\mu) = \frac{r(\lambda-\mu)^{bc}_{de}}{a(\lambda-\mu)}C_e(\mu)A_{ad}(\lambda) + \frac{b(\lambda-\mu)}{a(\lambda-\mu)}C_b(\mu)A_{ac}(\lambda), \tag{5.30}$$

$$B_c(\mu)A_{ab}(\lambda) = \frac{r(\lambda-\mu)^{de}_{bc}}{a(\lambda-\mu)}A_{db}(\lambda)B_e(\mu) + \frac{b(\lambda-\mu)}{a(\lambda-\mu)}A_{cb}(\mu)B_a(\lambda). \tag{5.31}$$

We do not write down the detailed proof here, since the procedure is similar to that of the previous propositions.

For the one-point correlation function associated with the fermion operators $(1-n_{\kappa,\downarrow})(1-n_{\kappa,\uparrow})$ and $S^z_\kappa$

$$F^{n_\kappa}_n(\{\mu_j\}, \xi_\kappa, \{\lambda_k\}) = \langle\Omega_N(\{\mu_j\}_{(p,n)})| \cdot (1-n_{\kappa,\downarrow})(1-n_{\kappa,\uparrow}) \cdot |\Omega_N(\{\lambda_k\}_{(p,n)})\rangle, \tag{5.32}$$

$$F^{S^z}_n(\{\mu_j\}, \xi_\kappa, \{\lambda_k\}) = \langle\Omega^p_N(\{\mu_j\}_{(p,n)})| \cdot S^z \cdot |\Omega_N(\{\lambda_k\}_{(p,n)})\rangle, \tag{5.33}$$

we have the following proposition:

**Proposition 6** *If both the Bethe state $|\Omega_N(\{u_j\})\rangle$ and the dual Bethe state $\langle\Omega_N(\{u_j\})|$ ($u_j = \lambda_j, \mu_j$) are eigenstates of the transfer matrix, then the correlation functions corresponding to the local fermion operators $(1-n_{\kappa,\downarrow})(1-n_{\kappa,\uparrow})$ and $S^z$ can be represented by*

$$F^{n_\kappa}_n(\{\mu_j\}_{(p,n)}, \xi_\kappa, \{\lambda_k\}_{(p,n)})$$
$$= (-1)^n \sum_{\sigma\in\mathcal{S}_n}\sum_{\sigma'\in\mathcal{S}_n} \phi_{\kappa-1}(\{\mu_j\})\phi^{-1}_\kappa(\{\lambda_k\})\,\mathcal{P}(1; \xi_\kappa; \{\mu_{\sigma'(j)}\}; \{\lambda_{\sigma(k)}\})$$

$$\tag{5.34}$$

$$F^{S^z}_n(\{\mu_j\}_{(p,n)}, \xi_\kappa, \{\lambda_k\}_{(p,n)})$$
$$= (-1)^n \sum_{\sigma\in\mathcal{S}_n}\sum_{\sigma'\in\mathcal{S}_n} \phi_{\kappa-1}(\{\mu_j\})\phi^{-1}_\kappa(\{\lambda_k\})$$

$$\times \sum_{i=1}^p \frac{b(\xi_\kappa, \lambda_{\sigma(i)})}{a(\xi_\kappa, \lambda_{\sigma(i)})} \prod_{k=p+1}^{i-1} \frac{c(\lambda_{\sigma(k)}, \lambda_{\sigma(i)})}{c(\lambda_{\sigma(k)}, \xi_\kappa)} \prod_{j=1,\neq i}^n \frac{c(\lambda_{\sigma(i)}, \lambda_{\sigma(j)})}{a(\lambda_{\sigma(i)}, \lambda_{\sigma(j)})} \prod_{\alpha=1}^N a(\lambda_{\sigma(i)}, \xi_\alpha)$$
$$\times P^L_n\left(\{\mu_{\sigma'(d)}\}_{(p,n)}; \{\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(i-1)}, \xi_\kappa, \lambda_{\sigma(i+1)}, \ldots, \lambda_{\sigma(n)}\}_{(p,n)}\right)$$
$$+ (-1)^n \sum_{\sigma\in\mathcal{S}_n}\sum_{\sigma'\in\mathcal{S}_n} \frac{1}{2}\phi_\kappa(\{\mu_{\sigma'(j)}\})\phi^{-1}_\kappa(\{\lambda_{\sigma(k)}\}) - \frac{1}{2}F^{n_\kappa}_n(\{\mu_j\}_{(p,n)}, \xi_\kappa, \{\lambda_k\}_{(p,n)}),$$

$$\tag{5.35}$$

*respectively, where*

$$\mathcal{P}(e; \delta; \{\mu_{\sigma'(j)}\}; \{\lambda_{\sigma(k)}\})$$

$$= \prod_{i=e}^{n} \frac{1}{a(\lambda_{\sigma(i)}, \delta)} P_n^L \left( \{\mu_{\sigma'(j)}\}_{(p,n)}; \{\lambda_{\sigma(k)}\}_{(p,n)} \right)$$

$$- \sum_{j=p+1}^{n} \frac{b(\lambda_{\sigma(j)}, \delta)}{a(\lambda_{\sigma(j)}, \delta)} \prod_{k=p+1}^{j-1} \frac{c(\lambda_{\sigma(k)}, \lambda_{\sigma(j)})}{c(\lambda_{\sigma(k)}, \delta)} \prod_{l=e, \neq j}^{n} \frac{1}{a(\lambda_{\sigma(l)}, \lambda_{\sigma(j)})}$$

$$\times P_n^L \left( \{\mu_{\sigma'(d)}\}_{(p,n)}; \{\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(e)}, \ldots, \lambda_{\sigma(j-1)}, \delta, \lambda_{\sigma(j+1)}, \ldots, \lambda_{\sigma(n)}\}_{(p,n)} \right)$$

$$- \sum_{i=e}^{p} \frac{b(\lambda_{\sigma(i)}, \delta)}{a(\lambda_{\sigma(i)}, \delta)} \prod_{k=1}^{i-1} \frac{c(\lambda_{\sigma(k)}, \lambda_{\sigma(i)})}{c(\lambda_{\sigma(k)}, \delta)} \prod_{j=1, \neq i}^{p} \frac{1}{a(\lambda_{\sigma(j)}, \lambda_{\sigma(i)})} \left[ \prod_{l=p+1}^{n} \frac{1}{a(\lambda_{\sigma(l)}, \lambda_{\sigma(i)})} \right.$$

$$\times P_n^L \left( \{\mu_{\sigma'(d)}\}_{(p,n)}; \{\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(e)}, \ldots, \lambda_{\sigma(i-1)}, \delta, \lambda_{\sigma(i+1)}, \ldots, \lambda_{\sigma(n)}\}_{(p,n)} \right)$$

$$- \sum_{l=p+1}^{n} \frac{b(\lambda_{\sigma(l)}, \lambda_{\sigma(i)})}{a(\lambda_{\sigma(l)}, \lambda_{\sigma(i)})} \prod_{m=p+1}^{l-1} \frac{c(\lambda_{\sigma(m)}, \lambda_{\sigma(l)})}{c(\lambda_{\sigma(m)}, \lambda_{\sigma(i)})} \prod_{q=p+1, \neq l}^{n} \frac{1}{a(\lambda_{\sigma(q)}, \lambda_{\sigma(l)})}$$

$$\times P_n^L \left( \{\mu_{\sigma'(d)}\}_{(p,n)}; \{\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(e)}, \ldots, \lambda_{\sigma(i-1)}, \delta, \lambda_{\sigma(i+1)}, \ldots, \right.$$

$$\left. \left. \lambda_{\sigma(l-1)}, \lambda_{\sigma(i)}, \lambda_{\sigma(l+1), \ldots, \lambda_{\sigma(n)}}\}_{(p,n)} \right) \right] \tag{5.36}$$

*Proof.* With the help of the commutation relations (5.25) and

$$D(\lambda) C_c(\mu) = \frac{1}{a(\mu, \lambda)} C_c(\mu) D(\lambda) - \frac{b(\mu, \lambda)}{a(\mu, \lambda)} C_c(\lambda) D(\mu), \tag{5.37}$$

by using the similar approach as before, one easily proves (5.34).

From (5.9), we have

$$S_\kappa^z = -\frac{1}{2} \prod_{j=1}^{\kappa-1} t(\xi_j) \cdot (A_{11}(\xi_\kappa) - A_{22}(\xi_\kappa)) \cdot \prod_{j=\kappa+1}^{N} t(\xi_j)$$

$$= \frac{1}{2} \prod_{j=1}^{\kappa-1} t(\xi_j) \cdot (t(\xi_\kappa) - D(\xi_\kappa) + 2A_{22}(\xi_\kappa)) \cdot \prod_{j=\kappa+1}^{N} t(\xi_j). \tag{5.38}$$

Substituting (5.38) into (5.33), we obtain

$$F_n^{Sz}(\{\mu_j\}_{(p,n)}, \xi_\kappa, \{\lambda_k\}_{(p,n)})$$

$$= \sum_{\sigma \in \mathcal{S}_n} \sum_{\sigma' \in \mathcal{S}_n} \phi_{\kappa-1}(\{\mu_j\}) \phi_\kappa^{-1}(\{\lambda_k\})$$

$$\times \langle \Omega_N(\{\mu_j\}_{(p,n)})| \cdot \left[ A_{22}(\xi_\kappa) + \frac{1}{2}(t(\xi_\kappa) - D(\xi_\kappa)) \right] \cdot |\Omega_N(\{\mu_j\}_{(p,n)})\rangle. \tag{5.39}$$

Then by using the commutation relations (5.30) and (5.25), one may prove that (5.39) gives rise to (5.35).

$\square$

## 5.2 two-point functions

In principle, by equations (5.1)-(5.9) with proper commutation relations derived from the GYBE, and similar method as that in the previous subsection, we may obtain any correlation function defined by (5.1).

As an example, in this subsection, we compute the correlation function associated with two adjacent fermion operators $(1 - n_{\kappa,\downarrow})c_{\kappa,\uparrow}^\dagger$ and $(1 - n_{\kappa+1,\downarrow})c_{\kappa+1,\uparrow}$. Considering the representations of the fermion operators (5.6) and (5.4), the correlation function is defined by

$$
\begin{aligned}
&F_n^{\uparrow\dagger,\uparrow}(\{\mu_j\}_{(p,n)}, \xi_\kappa, \xi_{\kappa+1}, \{\lambda_k\}_{(p,n)}) \\
&= \langle\Omega_N^p(\{\mu_j\})|(1 - n_{\kappa,\downarrow})c_{\kappa,\uparrow}^\dagger (1 - n_{\kappa+1,\downarrow})c_{\kappa+1,\uparrow}|\Omega_N^p(\{\lambda_k\})\rangle \\
&= \langle\Omega_N^p(\{\mu_j\})| \prod_{j=1}^{\kappa-1} t(\xi_j) \cdot C_2(\xi_\kappa)\, B_2(\xi_{\kappa+1}) \cdot \prod_{j=k+2}^{N} t(\xi_j)|\Omega_N^p(\{\lambda_k\})\rangle.
\end{aligned}
\tag{5.40}
$$

Here we have used the following property: for the supersymmetric $t$-$J$ model with periodic boundary condition, the transfer matrices satisfy the relation $\prod_{i=1}^{N} t(\lambda_i) = 1$. Then with the help of the commutation relations (5.24), (5.30), (5.37), and

$$
B_a(\lambda)C_b(\mu) = -C_b(\mu)B_a(\lambda) + \frac{b(\lambda,\mu)}{a(\lambda,\mu)}[D(\mu)A_{ab}(\lambda) - D(\lambda)A_{ab}(\mu)],
\tag{5.41}
$$

one proves the following proposition

**Proposition 7** *If both the Bethe state $|\Omega_N(\{u_j\})\rangle$ and the dual Bethe state $\langle\Omega_N(\{u_j\})|$ ($u_j = \lambda_j, \mu_j$) are eigenstates of the transfer matrix, then the two-point correlation functions associated with the local fermion operators $(1 - n_{\kappa,\downarrow})c_{\kappa,\uparrow}^\dagger$ and $(1 - n_{\kappa+1,\downarrow})c_{\kappa+1,\uparrow}$ can be represented by*

$$
\begin{aligned}
&F_n^{\uparrow\dagger,\uparrow}(\{\mu_j\}_{(p,n)}, \xi_\kappa, \xi_{\kappa+1}, \{\lambda_k\}_{(p,n)}) \\
&= \sum_{\sigma\in\mathcal{S}_n} \sum_{\sigma'\in\mathcal{S}_n} \phi_{\kappa-1}(\{\mu_j\})\phi_{\kappa+1}^{-1}(\{\lambda_k\}) \sum_{i=1}^{p} (-1)^{i-1} \frac{b(\xi_{\kappa+1}, \lambda_{\sigma(i)})}{a(\xi_{\kappa+1}, \lambda_{\sigma(i)})} \\
&\quad \times \Bigg\{ \sum_{j=i+1}^{p} \frac{b(\xi_{\kappa+1}, \lambda_{\sigma(j)})}{a(\xi_{\kappa+1}, \lambda_{\sigma(j)})} \prod_{l=i+1}^{j-1} \frac{c(\lambda_{\sigma(l)}, \lambda_{\sigma(j)})}{a(\lambda_{\sigma(l)}, \xi_\kappa)} \prod_{k=i+1, \neq j}^{p} \frac{c(\lambda_{\sigma(j)}, \lambda_{\sigma(k)})}{a(\lambda_{\sigma(j)}, \lambda_{\sigma(k)})} \prod_{\alpha=1}^{N} a(\lambda_{\sigma(j)}, \xi_\alpha) \\
&\qquad \times \mathcal{P}\left(i + 1; \lambda_{\sigma(i)}; \{\mu_{\sigma'(d)}\}_{(p,n)}; \{\lambda'_{\sigma(f)}\}_{(p,n)}\right) \\
&\quad - \prod_{j=i+1}^{p} \frac{c(\lambda_{\sigma(i)}, \lambda_{\sigma(j)})}{a(\lambda_{\sigma(i)}, \lambda_{\sigma(j)})} \prod_{\alpha=1}^{N} a(\lambda_{\sigma(i)}, \xi_\alpha)\, \mathcal{P}\left(i + 1; \xi_{\kappa+1}; \{\mu_{\sigma'(d)}\}_{(p,n)}; \{\lambda_{\sigma(f)}^*\}_{(p,n)}\right)
\end{aligned}
$$

$$-\sum_{j=i+1}^{p} \frac{b(\lambda_{\sigma(i)}, \lambda_{\sigma(j)})}{a(\lambda_{\sigma(i)}, \lambda_{\sigma(j)})} \prod_{l=i+1}^{j-1} \frac{c(\lambda_{\sigma(l)}, \lambda_{\sigma(j)})}{a(\lambda_{\sigma(l)}, \lambda_{\sigma(i)})} \prod_{k=i+1, \neq j}^{p} \frac{c(\lambda_{\sigma(j)}, \lambda_{\sigma(k)})}{a(\lambda_{\sigma(j)}, \lambda_{\sigma(k)})} \prod_{\alpha=1}^{N} a(\lambda_{\sigma(j)}, \xi_{\alpha})$$

$$\times \mathcal{P}\left(i+1; \xi_{\kappa+1}; \{\mu_{\sigma'(d)}\}_{(p,n)}; \{\lambda''_{\sigma(f)}\}_{(p,n)}\right)\Big\}, \tag{5.42}$$

where $\mathcal{P}$ is given by (5.36) and the spectral parameters $\lambda', \lambda^*$ and $\lambda''$ are given by

$$\lambda'_{\sigma(k)} = \begin{cases} \xi_{\kappa} & (k=1) \\ \lambda_{\sigma(k-1)} & (2 \leq k \leq i) \\ \lambda_{\sigma(k)} & (i+1 \leq k \leq n \text{ and } k \neq j) \\ \xi_{\kappa+1} & (k=j) \end{cases} , \lambda^*_{\sigma(k)} = \begin{cases} \xi_{\kappa} & (k=1) \\ \lambda_{\sigma(k-1)} & (k=2,\ldots,i) \\ \lambda_{\sigma(k)} & (i+1 \leq k \leq n) \end{cases} ,$$

$$\text{and} \qquad \lambda''_{\sigma(k)} = \begin{cases} \xi_{\kappa} & (k=1) \\ \lambda_{\sigma(k-1)} & (2 \leq k \leq i) \\ \lambda_{\sigma(k)} & (i+1 \leq k \leq n \text{ and } k \neq j) \\ \lambda_{\sigma(i)} & (k=j) \end{cases} ,$$

respectively.

# 6 Conclusion and discussion

In this paper, we constructed the determinant representations of scalar products and the correlation functions of the supersymmetric $t$-$J$ model with the help of the factorizing $F$-matrix. Because the $t$-$J$ model is an important model in the realm of the high $T_c$ superconductivity, we hope our results may enlarge the range of applications in this field. We also hope our results may offer a new understanding of the mathematical structures of the model. An interesting problem is to extend the results in this paper to the $t$-$J$ model with open boundary condition. This is under consideration and results will be reported elsewhere.

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